Squared Position Error Boundusage

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The received waveform at the agent from the anchor k can be written as

$$r_k(t) = \sum_{l=1}^{L_k} \alpha_k^{(l)} s(t - \tau_k^{(l)}) + z_k(t) \tag{1}$$

[1] proves that the Squared Position Error Bound of each position can be defined form the Fisher Information Matrix as

$$P(p) = \frac{c^2}{8\pi^2\beta^2} \frac{2\sum_{k\in N_L}(1-\chi_k)SNR_k^{(1)}}{\sum_{k\in N_L}\sum_{m\in N_L}(1-\chi_k)(1-\chi_m)SNR_k^{(1)}SNR_m^{(1)}\sin^2(\phi_k-\phi_m)} \tag{2}$$

where p is the position of the agent, k represent the k^{th} anchor, $SNR_k^{(1)}$ is the SNR of the first path and χ_k is determined only by the waveform $s(t)$ and the NLOS biases of the multi path component.

Assuming that χ_k and $SNR_k^{(1)}$ are all the same for all the anchors. χ_k is the same under the circumstance that $\tau_k^{(i)} - \tau_k^{(j)}$ and the number lines L_k are the same for all the anchors, which in other words the information is irrelative with the distance. To simplify the proof, we use χ and SNR to denote χ_k and $SNR_k^{(1)}$, and the size of N_L is denoted as N Thus we can get

$$P(p) = \frac{c^2}{8\pi^2\beta^2} \frac{2N(1-\chi)SNR}{(1-\chi)^2 SNR^2 \sum_{k\in N_L}\sum_{m\in N_L}\sin^2(\phi_k-\phi_m)} \tag{3}$$

Now to calculate the position error bound, we only need to focus on $\sum_{k\in N_L}\sum_{m\in N_L}\sin^2(\phi_k-\phi_m)$ and we use F to denote it.

Assume that all the anchors are laid out clockwise, which in another word AP_i is adjacent to AP_{i-1} and AP_{i+1}. Thus the layout has a property that

$$\phi_k - \phi_m = \phi_k - \phi_{k-1} + \phi_{k-1} - \phi_{k-2} \cdots + \phi_{m+1} - \phi_m \tag{4}$$

and to address the situation that when $k < m$, we can define that $\phi_k - \phi_m = \phi_k - \phi_{k-1} + \cdots + \phi_2 - \phi_1 + \phi_N - \phi_1 + \phi_N - \phi_{N-1} + \cdots + \phi_{m+1} - \phi_m$

Then use u_i to denote $\phi_{i+1} - \phi_i$ and specifically $u_n = \phi_N - \phi_1 + 2\pi \mod 2\pi$, which in other words represents the intersection angle between the adjacent APs. Thus the expression of object function can be simplified to:

$$\begin{aligned} F = \sum_{k \in N_L} \sum_{m \in N_L} \sin^2(\phi_k - \phi_m) = \sum_{i \in N_L} \sin^2(u_i) + \sum_{i \in N_L} \sin^2(u_i + u_{i+1}) + \\ \sum_{i \in N_L} \sin^2(u_i + u_{i+1} + u_{i+1}) + \cdots + \sum_{i \in N_L} \sin^2(u_i + u_{i+1} + \cdots + u_{i+N-1}) \end{aligned} \tag{5}$$

We need to emphasize one point that for the cases that when $i + k > N$, we define that $u_{i+k} = u_{i+k-N}$, for example $u_{N+2} = u_2$. To find minimum of the error bound $P(p)$, we need to maximize the object function F under a set of angles $\{u_1, u_2, \ldots, u_N\}$.

For N = 4, which there are four APs and $\{u_1, u_2, u_3, u_4\}$. Due to the geometric constraints, the goal is to calculate the maximum of F under the constraints that $u_1 + u_2 + u_3 + u_4 = 2\pi$. Applying the Lagrange Multiplier Approach, we can transfer the problem to solve the extreme point of the Lagrange function. The methods will be specified below:

$$\begin{aligned} L(u_1, u_2, u_3, u_4) &= F(u_1, u_2, u_3, u_4) + \lambda(u_1 + u_2 + u_3 + u_4 - 2\pi) \\ &= \sum_{i=1}^{4} \sin^2(u_i) + \sum_{i=1}^{4} \sin^2(u_i + u_{i+1}) + \sum_{i=1}^{4} \sin^2(u_i + u_{i+1} + u_{i+1}) \\ &+ \lambda(u_1 + u_2 + u_3 + u_4 - 2\pi) \end{aligned} \tag{6}$$

To calculate the maximum and the minimum of F, we need to take the derivative of each parameter u_i and the result is

$$\begin{cases} \frac{\partial L}{\partial u_1} = \sin(2u_1) + \sin(2(u_1+u_2)) + \sin(2(u_4+u_1)) + \sin(2(u_1+u_2+u_3)) \\ + \sin(2(u_3+u_4+u_1)) + \sin(2(u_4+u_1+u_2)) + \lambda = 0 \\ \frac{\partial L}{\partial u_2} = \sin(2u_2) + \sin(2(u_1+u_2)) + \sin(2(u_2+u_3)) + \sin(2(u_1+u_2+u_3)) \\ + \sin(2(u_2+u_3+u_4)) + \sin(2(u_4+u_1+u_2)) + \lambda = 0 \\ \frac{\partial L}{\partial u_3} = \sin(2u_3) + \sin(2(u_2+u_3)) + \sin(2(u_3+u_4)) + \sin(2(u_1+u_2+u_3)) \\ + \sin(2(u_3+u_4+u_1)) + \sin(2(u_2+u_3+u_4)) + \lambda = 0 \\ \frac{\partial L}{\partial u_4} = \sin(2u_4) + \sin(2(u_3+u_4)) + \sin(2(u_4+u_1)) + \sin(2(u_2+u_3+u_4)) \\ + \sin(2(u_3+u_4+u_1)) + \sin(2(u_4+u_1+u_2)) + \lambda = 0 \\ u_1+u_2+u_3+u_4-2\pi = 0 \end{cases} \tag{7}$$

subscribe the last equation into the first four equations, we can get

$$\begin{cases} \sin(2u_1) - \sin(2u_2) - \sin(2u_3) - \sin(2u_4) + \sin(2(u_1+u_2)) + \sin(2(u_4+u_1)) = -\lambda & (8a) \\ \sin(2u_2) - \sin(2u_1) - \sin(2u_3) - \sin(2u_4) + \sin(2(u_1+u_2)) + \sin(2(u_2+u_3)) = -\lambda & (8b) \\ \sin(2u_3) - \sin(2u_1) - \sin(2u_2) - \sin(2u_4) + \sin(2(u_2+u_3)) + \sin(2(u_3+u_4)) = -\lambda & (8c) \\ \sin(2u_4) - \sin(2u_1) - \sin(2u_2) - \sin(2u_3) + \sin(2(u_3+u_4)) + \sin(2(u_4+u_1)) = -\lambda & (8d) \end{cases}$$

The extreme point must satisfy the equation above. Equation(8a) minus equation(8c) is

$$2\sin(2u_1) - 2\sin(2u_3) + 2\sin(2(u_1+u_2)) - 2\sin(2(u_2+u_3)) = 0 \tag{9}$$

Similarly, equation(8b) minus equation(8d) is

$$2\sin(2u_2) - 2\sin(2u_4) + 2\sin(2(u_1+u_2)) - 2\sin(2(u_1+u_4)) = 0 \tag{10}$$

Simplify the two equations above,

$$\begin{cases} \cos(u_4)\cos(u_2)\sin(u_1-u_3) = 0 \\ \cos(u_1)\cos(u_3)\sin(u_2-u_4) = 0 \end{cases} \tag{11}$$

the results about the extreme point are analyzed under three different conditions.

1. Both the two corresponding sets of opposite angles equals each other. Then we have $u_1 = u_3$, $u_2 = u_4$.

2. only one set of opposite angles equals each other. For convenience, let $u_1 = u_3$ while $u_2 \neq u_3$. To satisfy the equation(11),

(a) if $u_4 = u_2 + \pi$, given that $2\pi = u_1 + u_2 + u_3 + u_4$, we have $u_1 + u_2 = \frac{\pi}{2}$.

(b) if $u_4 \neq u_2 + \pi$, then we have $u_1 = u_3 = \frac{\pi}{2}$.

3. neither of the two sets of opposite angles equals each other, then we have $u_1 = u_2 = \frac{\pi}{2}$.

However, as for the third condition, the results cannot both satisfy the equation(8a), (8b), (8c),(8d).

All the possible extreme points has been discussed above. Therefore, there are only three possible types of the extreme points.

The extreme point is for u_i which is defined as $\phi_{i+1} - \phi_i$. The situation we consider is that there are four APs in the field. The extreme points are suitable for all the situations for the quadrangle, but not all the quadrangle can achieve the extreme point and from the discussion below only the rectangle and the quadrangle which four vertex are in the same circle.

The first type is the intersections of diagonal lines of the quadrangle corresponding to the condition(1), the second type lies in the circumscribed circle of a rectangle because $u_1 + u_2 = \frac{\pi}{2}$ and the last type is the intersections of two circles, whose diameters are the two long opposite sides.

Especially, when the quadrangle is a rectangle, the first type of the extreme points lie in the center of the rectangle, as shown in the figure 1 .

However, not all the center points in the rectangle possess the property. Under some conditions, they appear to be saddle points rather than the extreme points.

As for the second type, the extreme points lie in the circumscribed circle of the rectangle, as shown in the figure 2.

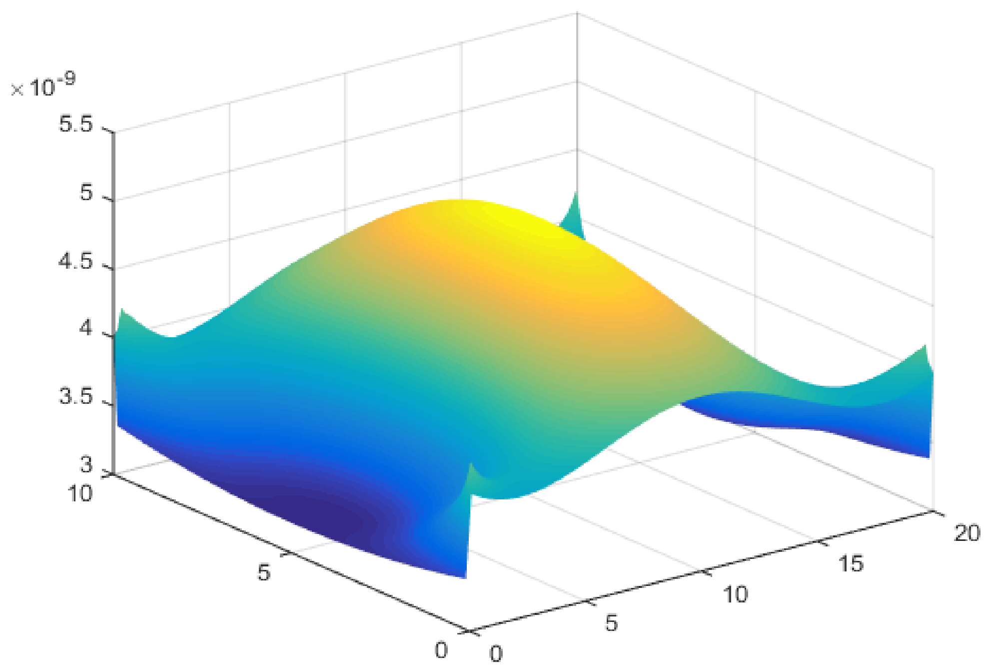

Figure 1: extreme point in the rectangle

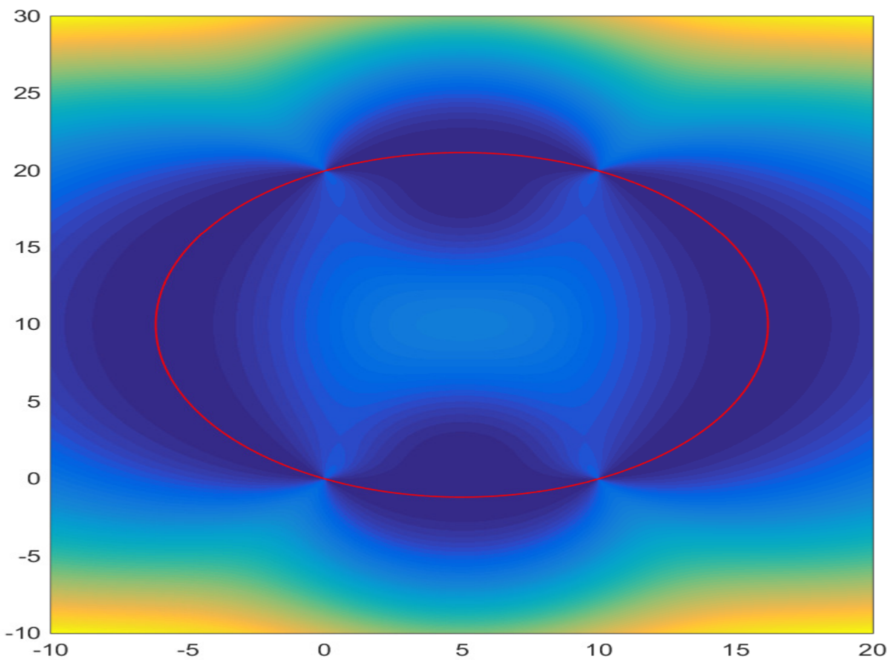

Figure 2: extreme point in the circumscribed circle

Finally, the last type of extreme points in the rectangle can be shown in the figure 2

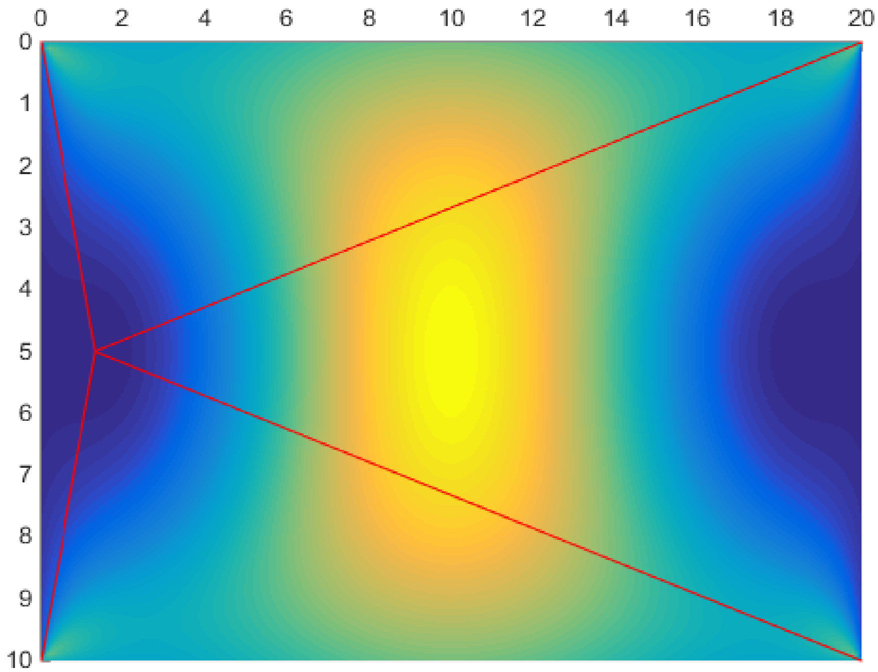

Figure 3: extreme point in the center

References

[1] Yuan Shen and Moe Z Win. Fundamental limits of wideband localization-part i: A general framework. *Information Theory, IEEE Transactions on*, 56(10):4956–4980, 2010.