Due: 2022/3/12

# Homework 4 - Typed

∗ If there is any problem, please contact TA.

Name:______________ Student ID:______________ Email: __________________

**Problem 1.** (50 points)
Given the definition of *pred n* (predecessor of n):

$$pred = \lambda n.\lambda f.\lambda x.n\ (\lambda g.\lambda h.h\ (g\ f))\ (\lambda u.x)\ (\lambda u.u)$$

Note that with such definition we don't have numbers less than zero.(Try pred 0 for example)
Please define following terms using lambda calculus:

1. sub m n (subtraction)
2. iszero n
3. leq m n (m is less or equal than n)
4. equal m n
5. factorial n (You can try to define it using pair)

(You can directly use the definition in the slides and the last homework, like add, tru, etc.)

*Solution.*

1. subtraction: $\lambda xy.y\ pred\ x$
2. iszero: $\lambda x.x\ (\lambda y.fls)\ tru$
3. leq: $\lambda xy.iszero(sub\ x\ y)$
4. equal: $\lambda xy.and\ (leq\ x\ y)\ (leq\ y\ x)$
5. $n! =$ if $n \leq 1$ then 1 else $n * (n-1)!$
   (a) Use pair to define factorial

   $$\begin{aligned} zz &= pair\ 1\ 1 \\ ss &= \lambda p.pair\ (multi\ (fst\ p)\ (snd\ p))\ (add\ (snd\ p)\ 1) \\ factorial &= \lambda x.fst\ (x\ ss\ zz) \end{aligned}$$

   pred can also be defined by pair!

   (b) factorial: $(\lambda y.y\ y)\ (\lambda fn.(iszero\ n)\ 1\ (times\ n\ (f\ f\ (pred\ n))))$

$\square$

**Problem 2.** (20 points) Prove the exchange lemma: If $\Gamma, x : t_1, y : t_2, \Gamma' \vdash e : t$, then $\Gamma, y : t_2, x : t_1, \Gamma' \vdash e : t$. (proof by induction on derivation of $\Gamma, x : t_1, y : t_2, \Gamma' \vdash e : t$)

*Proof.* By induction on derivation of $\Gamma, x : t_1, y : t_2, \Gamma' \vdash e : t$

1. case$\frac{x:t\in\Gamma}{\Gamma\vdash x:t}$

   Need to prove: If $\Gamma, x : t_1, y : t_2, \Gamma' \vdash x_1 : t$, then $\Gamma, y : t_2, x : t_1, \Gamma' \vdash x_1 : t$.

   (1) $x_1 : t \in \Gamma, x : t_1, y : t_2, \Gamma'$ (by assumption)

   (2) $x_1 : t \in \Gamma, y : t_2, x : t_1, \Gamma'$ (by (1))

   (3) $\Gamma, y : t_2, x : t_1, \Gamma' \vdash x_1 : t$. (by (2) and T-Var)

2. case $\frac{}{\Gamma\vdash \mathit{true}:\mathit{bool}}$

   Need to prove: If $\Gamma, x : t_1, y : t_2, \Gamma' \vdash \mathit{true} : t$, then $\Gamma, y : t_2, x : t_1, \Gamma' \vdash \mathit{true} : t$.

   (1) $\Gamma, y : t_2, x : t_1, \Gamma' \vdash \mathit{true} : t$. (by T-True)

3. case $\frac{}{\Gamma\vdash \mathit{false}:\mathit{bool}}$

   Need to prove: If $\Gamma, x : t_1, y : t_2, \Gamma' \vdash \mathit{false} : t$, then $\Gamma, y : t_2, x : t_1, \Gamma' \vdash \mathit{false} : t$.

   (1) $\Gamma, y : t_2, x : t_1, \Gamma' \vdash \mathit{false} : t$. (by T-False)

4. case $\frac{\Gamma\vdash e_1:\mathit{bool} \quad \Gamma\vdash e_2:t \quad \Gamma\vdash e_3:t}{\Gamma\vdash \mathit{if}\ e_1\ \mathit{then}\ e_2\ \mathit{else}\ e_3:t}$

   Need to prove: If $\Gamma, x : t_1, y : t_2, \Gamma' \vdash \mathit{if}\ e_1\ \mathit{then}\ e_2\ \mathit{else}\ e_3 : t$, then $\Gamma, y : t_2, x : t_1, \Gamma' \vdash \mathit{if}\ e_1\ \mathit{then}\ e_2\ \mathit{else}\ e_3 : t$.

   (1)

   $\Gamma, x : t_1, y : t_2, \Gamma' \vdash e_1 : \mathit{bool}$,

   $\Gamma, x : t_1, y : t_2, \Gamma' \vdash e_2 : t$,

   $\Gamma, x : t_1, y : t_2, \Gamma' \vdash e_3 : t$ (by assumption)

   (2)

   $\Gamma, y : t_2, x : t_1, \Gamma' \vdash e_1 : \mathit{bool}$,

   $\Gamma, y : t_2, x : t_1, \Gamma' \vdash e_2 : t$,

   $\Gamma, y : t_2, x : t_1, \Gamma' \vdash e_3 : t$ (by (1) and I.H.)

   (3) $\Gamma, y : t_2, x : t_1, \Gamma' \vdash \mathit{if}\ e_1\ \mathit{then}\ e_2\ \mathit{else}\ e_3 : t$. (by (2) and T-If)

5. case $\frac{\Gamma,x:t_1\vdash e_2:t_2}{\Gamma\vdash\lambda x:t_1.e_2:t_1\to t_2}$

   Need to prove: If $\Gamma, x : t_3, y : t_4, \Gamma' \vdash \lambda x : t_1.e_2 : t_1 \to t_2$, then $\Gamma, y : t_4, x : t_3, \Gamma' \vdash \lambda x : t_1.e_2 : t_1 \to t_2$.

   (1) $\Gamma, x : t_3, y : t_4, \Gamma' \vdash e_2 : t_2$ (by assumption)

   (2) $\Gamma, y : t_4, x : t_3, \Gamma' \vdash e_2 : t_2$ (by (1) and I.H.)

   (3) $\Gamma, y : t_4, x : t_3, \Gamma' \vdash \lambda x : t_1.e_2 : t_1 \to t_2$. (by (2) and T-Abs)

6. case $\frac{\Gamma \vdash e_1 : t_{11} \to t_{12} \quad \Gamma \vdash e_2 : t_{11}}{\Gamma \vdash e_1\ e_2 : t_{12}}$

   Need to prove: If $\Gamma, x : t_1, y : t_2, \Gamma' \vdash e_1\ e_2 : t_{12}$, then $\Gamma, y : t_2, x : t_1, \Gamma' \vdash e_1\ e_2 : t_{12}$.

   (1)

   $\Gamma, x : t_1, y : t_2, \Gamma' \vdash e_1 : t_{11} \to t_{12}$,

   $\Gamma, x : t_1, y : t_2, \Gamma' \vdash e_2 : t_{11}$ (by assumption)

   (2)

   $\Gamma, y : t_2, x : t_1, \Gamma' \vdash e_1 : t_{11} \to t_{12}$,

   $\Gamma, y : t_2, x : t_1, \Gamma' \vdash e_2 : t_{11}$ (by (1) and I.H.)

   (3) $\Gamma, y : t_2, x : t_1, \Gamma' \vdash e_1\ e_2 : t_{12}$. (by (2) and T-App)

□

**Problem 3.** (20 points) Prove the **weakening lemma**: If $\Gamma \vdash e : t$ then $\Gamma, x : t' \vdash e : t$ (provided x not in Dom(Γ))

*Proof.* by induction on derivation of $\Gamma \vdash e : t$

1. case$\frac{x:t \in \Gamma}{\Gamma \vdash x:t}$

   Need to prove: If $\Gamma \vdash x : t$ then $\Gamma, y : t' \vdash x : t$

$$(1) x : t \in \Gamma \qquad (by\ assumption)$$
$$(2) x : t \in \Gamma, y : t' \qquad (by\ (1))$$
$$(3) \Gamma, y : t' \vdash e : t \qquad (by\ (2)\ and\ T-Var)$$

2. case $\frac{}{\Gamma \vdash true : bool}$

   Need to prove: If $\Gamma \vdash true : bool$ then $\Gamma, y : t' \vdash true : bool$

$$(1)\Gamma, x : t' \vdash true : bool \qquad (T-True)$$

3. case $\frac{}{\Gamma \vdash false : bool}$

   Need to prove: If $\Gamma \vdash false : bool$ then $\Gamma, y : t' \vdash false : bool$

$$(1)\Gamma, x : t' \vdash false : bool \qquad (T-False)$$

4. case $\frac{\Gamma \vdash e_1 : bool \quad \Gamma \vdash e_2 : t \quad \Gamma \vdash e_3 : t}{\Gamma \vdash if\ e_1\ then\ e_2\ else\ e_3 : t}$

   Need to prove: If $\Gamma \vdash if\ e_1\ then\ e_2\ else\ e_3 : t$ then $\Gamma, y : t' \vdash if\ e_1\ then\ e_2\ else\ e_3 : t$

$$\begin{aligned} &(1)\Gamma \vdash e_1 : bool, \\ &\quad \Gamma \vdash e_2 : t, \\ &\quad \Gamma \vdash e_3 : t && (by\ assumption) \\ &(2)\Gamma, y : t' \vdash e_1 : bool, \\ &\quad \Gamma, y : t' \vdash e_2 : t, \\ &\quad \Gamma, y : t' \vdash e_3 : t && (by\ (1)\ and\ I.H.) \\ &(3)\Gamma, y : t' \vdash if\ e_1\ then\ e_2\ else\ e_3 : t && (by\ (2)\ and\ T-If) \end{aligned}$$

5. case $\frac{\Gamma, x:t_1 \vdash e_2 : t_2}{\Gamma \vdash \lambda x:t_1.e_2 : t_1 \to t_2}$

   Need to prove: If $\Gamma \vdash \lambda x : t_1.e_2 : t_1 \to t_2$ then $\Gamma, y : t' \vdash \lambda x : t_1.e_2 : t_1 \to t_2$

$$
\begin{aligned}
&(1)\Gamma, x : t_1 \vdash e_2 : t_2 && (\textit{by assumption})\\
&(2)\Gamma, x : t_1, y : t' \vdash e_2 : t_2 && (\textit{by } (1) \textit{ and } I.H)\\
&(3)\Gamma, y : t', x : t_1 \vdash e_2 : t_2 && (\textit{by exchange lemma})\\
&(4)\Gamma, y : t' \vdash \lambda x : t_1.e_2 : t_1 \to t_2 && (\textit{by } T - Abs)
\end{aligned}
$$

6. case $\frac{\Gamma \vdash e_1 : t_{11} \to t_{12} \quad \Gamma \vdash e_2 : t_{11}}{\Gamma \vdash e_1 \ e_2 : t_{12}}$

   Need to prove: If $\Gamma \vdash e_1 \ e_2 : t_{12}$ then $\Gamma, y : t' \vdash e_1 \ e_2 : t_{12}$

$$
\begin{aligned}
&(1)\Gamma \vdash e_1 : t_{11} \to t_{12}, &&\\
&\quad \Gamma \vdash e_2 : t_{11} && (\textit{by assumption})\\
&(2)\Gamma, y : t' \vdash e_1 : t_{11} \to t_{12}, &&\\
&\quad \Gamma, y : t' \vdash e_2 : t_{11} && (\textit{by } (1) \textit{ and } I.H)\\
&(3)\Gamma, y : t' \vdash e_1 \ e_2 : t_{12} && (\textit{by } (2) \textit{ and } T - App)
\end{aligned}
$$

□

**Problem 4.** (20 points)
Prove the **substitution lemma**: If $\Gamma, x : t' \vdash e : t$ and $\Gamma \vdash v : t'$ then $\Gamma \vdash e[v/x] : t$.

*Proof.* by induction on the derivation of $\Gamma \vdash e : t$

1. case$\frac{x:t \in \Gamma}{\Gamma \vdash x:t}$

   Need to prove: If $\Gamma, x : t' \vdash y : t$, and $\Gamma \vdash v : t'$, then $\Gamma \vdash y[v/x] : t$.

   If $y == x$:

$$
\begin{aligned}
&(1)y[v/x] = v &&\\
&(2)\Gamma \vdash y[v/x] : t && (\textit{by } (1) \textit{ and assumption})
\end{aligned}
$$

   If $y \neq x$:

$$
\begin{aligned}
&(1)y[v/x] = y &&\\
&(2)\Gamma, x : t' \vdash y : t && (\textit{by assumption})\\
&(3)y : t \in \Gamma, x : t' && (\textit{by inversion of } T - Var)\\
&(4)y : t \in \Gamma &&\\
&(5)\Gamma \vdash y : t && (\textit{by } T - Var)\\
&(6)\Gamma \vdash y[v/x] : t && (\textit{by } (1) \textit{ and } (5))
\end{aligned}
$$

2. case $\frac{}{\Gamma \vdash true : bool}$

   Need to prove: If $\Gamma, x : t' \vdash true : bool$, and $\Gamma \vdash v : t'$, then $\Gamma \vdash true[v/x] : bool$.

$$(1) \Gamma \vdash true[v/x] : bool \qquad (T - True)$$

3. case $\frac{}{\Gamma \vdash false : bool}$

   Need to prove: If $\Gamma, x : t' \vdash false : bool$, and $\Gamma \vdash v : t'$, then $\Gamma \vdash false[v/x] : bool$.

$$(1) \Gamma \vdash false[v/x] : bool \qquad (T - False)$$

4. case $\frac{\Gamma \vdash e_1 : bool \quad \Gamma \vdash e_2 : t \quad \Gamma \vdash e_3 : t}{\Gamma \vdash if\ e_1\ then\ e_2\ else\ e_3 : t}$

   Need to prove: If $\Gamma, x : t' \vdash if\ e_1\ then\ e_2\ else\ e_3 : t$, and $\Gamma \vdash v : t'$, then $\Gamma \vdash (if\ e_1\ then\ e_2\ else\ e_3)[v/x] : t$.

$$\begin{array}{ll}
(1) \Gamma, x : t' \vdash e_1 : bool, & \\
\quad \Gamma, x : t' \vdash e_2 : t, & \\
\quad \Gamma, x : t' \vdash e_3 : t & (by\ assumption) \\
(2) \Gamma \vdash e_1[v/x] : bool, & \\
\quad \Gamma \vdash e_2[v/x] : t, & \\
\quad \Gamma \vdash e_3[v/x] : t & (by\ (1)\ and\ I.H.) \\
(3) \Gamma \vdash (if\ e_1\ then\ e_2\ else\ e_3)[v/x] : t & (by\ (2)\ and\ T - If)
\end{array}$$

5. case $\frac{\Gamma, x : t_1 \vdash e_2 : t_2}{\Gamma \vdash \lambda x : t_1 . e_2 : t_1 \to t_2}$

   Need to prove: If $\Gamma, y : t' \vdash \lambda x : t_1.e_2 : t_1 \to t_2$, and $\Gamma \vdash v : t'$, then $\Gamma \vdash (\lambda x : t_1.e_2)[v/y] : t_1 \to t_2$.

$$\begin{array}{ll}
(1) \Gamma, y : t', x : t_1 \vdash e_2 : t_2 & (by\ assumption) \\
(2) \Gamma, x : t_1, y : t' \vdash e_2 : t_2 & (by\ exchange\ lemma) \\
(3) \Gamma \vdash v : t' & (by\ assumption) \\
(4) \Gamma, x : t_1 \vdash v : t' & (by\ (3)\ and\ weakening\ lemma) \\
(5) \Gamma, x : t_1 \vdash e_2[v/y] : t_2 & (by\ (2), (3)\ and\ I.H.) \\
(6) \Gamma \vdash (\lambda x : t_1.e_2)[v/y] : t_1 \to t_2 & (by\ (5)\ and\ T - Abs)
\end{array}$$

6. case $\frac{\Gamma \vdash e_1 : t_{11} \to t_{12} \quad \Gamma \vdash e_2 : t_{11}}{\Gamma \vdash e_1\ e_2 : t_{12}}$

   Need to prove: If $\Gamma, x : t' \vdash e_1\ e_2 : t_{12}$, and $\Gamma \vdash v : t'$, then $\Gamma \vdash (e_1\ e_2)[v/x] : t_{12}$.

$$\begin{array}{ll}
(1) \Gamma, x : t' \vdash e_1 : t_{11} \to t_{12}, & \\
\quad \Gamma, x : t' \vdash e_2 : t_{11} & (by\ assumption) \\
(2) \Gamma \vdash v : t' & (by\ assumption) \\
(3) \Gamma \vdash e_1[v/x] : t_{11} \to t_{12}, & \\
\quad \Gamma \vdash e_2[v/x] : t_{11} & (by\ (1), (2)\ and\ I.H) \\
(3) \Gamma \vdash (e_1\ e_2)[v/x] : t_{12} & (by\ (3)\ and\ T - App)
\end{array}$$

□