Due: 2022/3/5

Homework 3 - Lambda

∗ If there is any problem, please contact TA.

Name:____________ Student ID:____________ Email: ________________

Problem 1. (40 points) Evaluate the following λ expressions using call-by-value and call-by-name. Show the complete steps of evaluation.

(a) $(\lambda x.((\lambda y.\ x + z + 3)\ 3)\ 5)$

(b) $((\lambda v.(\lambda w.w))\ ((\lambda x.x)\ (y\ (\lambda z.z))))$

(c) $((\lambda x.\ x\ x)\ (\lambda y.\ y\ y))$

(d) $((\lambda x.\lambda y.x)\ (\lambda z.z\ \lambda u.u))$

Solution.

(a). Call-by-value and call-by-name:

$$\begin{aligned}&(\lambda x.((\lambda y.\ x + z + 3)\ 3)\ 5)\\ \rightarrow&(\lambda x.((\lambda y.\ x + z + 3)\ 3)\ 5)\end{aligned}$$

(b). Call-by-value:

$$\begin{aligned}&(\lambda v.(\lambda w.w))\ ((\lambda x.x)\ (y\ (\lambda z.z)))\\ \rightarrow&(\lambda v.(\lambda w.w))\ ((\lambda x.x)\ (y\ (\lambda z.z)))\\ \rightarrow&(\lambda v.(\lambda w.w))\ ((\lambda x.x)\ (y\ (\lambda z.z)))\end{aligned}$$

(No reduction can be applied.)

Call-by-name:

$$\begin{aligned}&(\lambda v.(\lambda w.w))\ ((\lambda x.x)\ (y\ (\lambda z.z)))\\ \rightarrow&\lambda w.w\end{aligned}$$

(c). Call-by-value and call-by-name:

$$\begin{aligned}&((\lambda x.\ x\ x)\ (\lambda y.\ y\ y))\\ \rightarrow&((\lambda y.\ y\ y)\ (\lambda y.\ y\ y))\\ \rightarrow&\ldots\end{aligned}$$

(d). Call-by-value and call-by-name:

$$\begin{aligned} &((\lambda x.\lambda y.x)\ (\lambda z.z\ \lambda u.u)) \\ \to &\lambda y.(\lambda z.z\ \lambda u.u) \end{aligned}$$

$\square$

Problem 2. (30 points)
Prove by induction: If e_1 is closed and $e_1 \to e_2$, then e_2 is closed. Suppose using call by value evaluation.

- Given the following definitions:
    1. Rules of free variables (You can use these rules directly without writing "By ...")

$$\frac{}{FV(x) = \{x\}} \qquad \frac{FV(e_1) = S_1 \quad FV(e_2) = S_2}{FV(e_1\ e_2) = S_1 \cup S_2} \qquad \frac{FV(e) = S}{FV(x.e) = S - \{x\}}$$

    2. Judgment form: **define** $e_1 \to e_2$

$$\frac{}{(\lambda x.e)\ v \to e[v/x]} \qquad \frac{e_1 \to e_1^{'}}{e_1\ e_2 \to e_1^{'}\ e_2} \qquad \frac{e_2 \to e_2^{'}}{v\ e_2 \to v\ e_2^{'}}$$

    3. Judgment form: **define** $e_1 \to^* e_2$

$$\frac{}{e_1 \to^* e_1} \qquad \frac{e_1 \to e_2 \qquad e_2 \to^* e_3}{e_1 \to^* e_3}$$

- And given this lemma:

    Lemma 1. $FV(e_1[e_2/x]) \subseteq (FV(e_1) - \{x\}) \cup FV(e_2)$

    You can use this lemma directly. (Proof is in the appendix)

Proof. By induction on the derivation of $e_1 \to e_2$

1. case $\frac{}{(\lambda x.e)\ v \to e[v/x]}$

   Need to prove: If $(\lambda x.e)\ v$ is closed and $(\lambda x.e)\ v \to e[v/x]$, then $e[v/x]$ is closed.

   (1) $FV((\lambda x.e)\ v) = \emptyset$ (by assumption)

   (2) $FV(e[v/x]) \subseteq (FV(e) - \{x\}) \cup FV(v)$ (by lemma 1)

   (3) $FV((\lambda x.e)\ v) = (FV(e) - \{x\}) \cup FV(v)$

   (4) $FV(e[v/x]) = \emptyset$ (by (1), (2) and (3))

2. case $\frac{e_1 \to e_1'}{e_1\ e_2 \to e_1'\ e_2}$

   Need to prove: If $e_1\ e_2$ is closed and $e_1\ e_2 \to e_1'\ e_2$, then $e_1'\ e_2$ is closed.

   (1) $FV(e_1\ e_2) = \emptyset$ (by assumption)

   (2) $FV(e_1'\ e_2) = FV(e_1') \cup FV(e_2)$

   (3) $FV(e_1\ e_2) = FV(e_1) \cup FV(e_2)$

   (4) $FV(e_1) = \emptyset$ and $FV(e_2) = \emptyset$ (by (1) and (3))

   (5) $FV(e_1') = \emptyset$ (by I.H.)

   (6) $FV(e_1'\ e_2) = \emptyset$ (by (2), (4) and (5))

3. case $\frac{e_2 \to e_2'}{v\ e_2 \to v\ e_2'}$

   Need to prove: If $v\ e_2$ is closed and $v\ e_2 \to v\ e_2'$, then $v\ e_2'$ is closed.

   (1) $FV(v\ e_2) = \emptyset$ (by assumption)

   (2) $FV(v\ e_2) = FV(v) \cup FV(e_2)$

   (3) $FV(v) = \emptyset$ and $FV(e_2) = \emptyset$ (by (1) and (2))

   (4) $FV(e_2') = \emptyset$ (by (3) and I.H.)

   (5) $FV(v\ e_2') = FV(v) \cup FV(e_2') = \emptyset$ (by (3) and (4))

$\square$

Problem 3. (30 points) Church encoding is a means of embedding data and operators into the λ calculus, the most familiar form being the Church numerals, a representation of the natural numbers using λ notation. Church numerals $\mathbf{0}, \mathbf{1}, \mathbf{2}, ...$, are defined as follows:

$$
\begin{aligned}
\mathbf{0} &= \lambda f.\lambda x.\ x \\
\mathbf{1} &= \lambda f.\lambda x.\ f\ x \\
\mathbf{2} &= \lambda f.\lambda x.\ f\ (f\ x) \\
\mathbf{3} &= \lambda f.\lambda x.\ f\ (f\ (f\ x)) \\
&\dots \\
\mathbf{n} &= \lambda f.\lambda x.\ f^n\ x \\
&\dots
\end{aligned}
$$

Church numerals takes two parameters f and x. Church numerals n means apply f to x n times. (You can refer to Wikipedia or other references about Church encoding to know more about the idea of church encoding)

(a) Define addition in λ calculus, and then show the evaluation of $3 + 2$.

(b) Define multiplication in λ calculus (Hint: use definition of addition), and then show the evaluation of 3×2.

(c) Give a definition of multiplication on Church numerals without using addition.

Solution.
(If you use a call-by-value evaluation, you can't reduce inside abstraction. You can try to pass two parameters and verify after reduction.)

(a)

$$add = \lambda a.\lambda b.\lambda f.\lambda x.(a\ f)\ (b\ f\ x)$$

$$\begin{aligned}
&3+2\\
=&add\ 3\ 2\\
=&(\lambda a.\lambda b.\lambda f.\lambda x.(a\ f)\ (b\ f\ x))\ 3\ 2\\
\rightarrow&(\lambda b.\lambda f.\lambda x.(3\ f)\ (b\ f\ x))\ 2\\
\rightarrow&\lambda f.\lambda x.(3\ f)\ (2\ f\ x)\\
=&\lambda f.\lambda x.((\lambda f.\lambda x.\ f^3\ x)\ f)\ ((\lambda f.\lambda x.\ f^2\ x)\ f\ x)\\
\rightarrow^*&\lambda f.\lambda x.(\lambda x.\ f^3\ x)\ (f^2\ x)\\
\rightarrow&\lambda f.\lambda x.f^5\ x
\end{aligned}$$

(b)

$$multi = \lambda m.\lambda n.m\ (add\ n)\ 0$$

$$\begin{aligned}
&3\times 2\\
=&multi\ 3\ 2\\
=&(\lambda m.\lambda n.m\ (add\ n)\ 0)\ 3\ 2\\
\rightarrow&(\lambda n.3\ (add\ n)\ 0)\ 2\\
\rightarrow&3\ (add\ 2)\ 0\\
=&(\lambda f.\lambda x.\ f\ (f\ (f\ x)))\ (add\ 2)\ 0\\
\rightarrow^*&add\ 2\ (add\ 2\ (add\ 2\ 0))\\
\rightarrow&add\ 2\ (add\ 2\ 2)\\
\rightarrow&add\ 2\ 4\\
\rightarrow&6
\end{aligned}$$

(c)

$$multi = \lambda m.\lambda n.\lambda f.m\ (n\ f)$$

$$
\begin{aligned}
& 3 \times 2 \\
=& \textit{multi } 3\ 2 \\
=& (\lambda m.\lambda n.\lambda f.m\ (n\ f))\ 3\ 2 \\
\rightarrow^{*}& \lambda f.3\ (2\ f) \\
\rightarrow^{*}& \lambda f.\lambda x.(2\ f)^{3}\ x \\
\rightarrow^{*}& \lambda f.\lambda x.(2\ f)^{2}\ (f^{2}\ x) \\
\rightarrow^{*}& \lambda f.\lambda x.2\ f\ (f^{4}\ x) \\
\rightarrow^{*}& \lambda f.\lambda x.f^{6}\ x
\end{aligned}
$$

$\square$