Due: 2022/2/26 23:59

Homework 2 - Inductive Proof

$*$ If there is any problem, please contact TA:yvonne_huang@sjtu.edu.cn..

Name:______________ Student ID:______________ Email: ___________________

Problem 1. (30 points)

(a) Please look at page 21 in slide "inductive-proof". In the proof of the second case $\frac{n\ nat}{S(n)\ nat}$, what is the assumption in this case and what is the difference between assumption and I.H.?

(b) We define a judgment form $IsNat\ x\ a$.

$$\frac{x\ nat}{IsNat\ x\ true}NatRule \qquad \frac{x\ list}{IsNat\ x\ false}ListRule \qquad \frac{x\ tree}{IsNat\ x\ false}TreeRule$$

For which rule we can use its inversion rule? If there exists such rule, point it out and give an explanation. If no rules can be inverted, give an explanation.

(c) We define a judgment form $add'\ n_1\ n_2\ n_3$ (another definition for addition):

$$\frac{}{add'\ Z\ Z\ Z}add'Z \qquad \frac{add'\ n_1\ n_2\ n_3}{add'\ (Sn_1)\ n_2\ (Sn_3)}add'-l \qquad \frac{add'\ n_1\ n_2\ n_3}{add'\ n_1\ (Sn_2)\ (Sn_3)}add'-r$$

For which rule we can use its inversion rule? If there exists such rule, point it out and give an explanation. If no rules can be inverted, give an explanation.

Solution.

(a) "n nat" is assumption. Assumption includes the judgments which are already given. I.H. means we can assume that the property P which we want to prove holds for a particular instance.

(b) We can only use the inversion rule of NatRule. Because the judgment "IsNat x true" can only come from NatRule while "IsNat x false" can comes from ListRule or TreeRule.

(c) No inversion rule of theses rules can be used.

add'Z is an axiom and we can derive nothing.

For *add'-l* and *add'-r*, consider the judgement "$add'\ (Sn_1)\ (Sn_2)\ (Sn_3)$". It can either come from "$add'\ n_1\ (Sn_2)\ n_3$" with *add'-l* or from "$add'\ (Sn_1)\ n_2\ n_3$" with *add'-r*.

Problem 2. (30 points)

(a) (15 points) Give an inductive definition of the judgment form max $n_1\ n_2\ n_3$, which indicates the max number between n_1 and n_2 is n_3.

(b) (15 points) Prove by induction: if max n_1 n_2 n_3, then max n_2 n_1 n_3.

Solution.

(a) **define** $max\ n_1\ n_2\ n_3$

$$\frac{}{max\ Z\ n\ n}maxZN \qquad \frac{}{max\ n\ Z\ n}maxNZ$$

$$\frac{max\ n_1\ n_2\ n_3}{max\ S(n_1)\ S(n_2)\ S(n_3)}maxS$$

(In maxZN, you can define it as $\frac{n\ nat}{max\ Z\ n\ n}$. Type check is not necessary.)

(b) **Proof:** By induction on the derivation of $max\ n_1\ n_2\ n_3$

1. **Case:** $\frac{}{\text{max}\ Z\ n\ n}maxZN$
   Need to Prove: if $max\ Z\ n\ n$, then $max\ n\ Z\ n$.

   $$(1) max\ n\ Z\ n \qquad (By\ maxNZ)$$

2. **Case:** $\frac{}{\text{max}\ Z\ n\ n}maxZN$
   Similar to the previous case.
3. **Case:** $\frac{max\ n_1\ n_2\ n_3}{max\ S(n_1)\ S(n_2)\ S(n_3)}maxS$
   Need to Prove: if $max\ S(n_1)\ S(n_2)\ S(n_3)$ then $max\ S(n_2)\ S(n_1)\ S(n_3)$

   $$\begin{aligned} &(1)\ max\ n_1\ n_2\ n_3 && (By\ assumption) \\ &(2)\ max\ n_2\ n_1\ n_3 && (By\ (1)\ and\ I.H.) \\ &(3)\ max\ S(n_2)\ S(n_1)\ S(n_3) && (By\ (2)\ and\ maxS) \end{aligned}$$

   (**Usually we can omit "let xx=..."**)

□

Problem 3. (20 points)

(a) Recall the definition of addition by $add\ n_1\ n_2\ n_3$ judgment taught in the lecture.

(b) (15 points) Prove by induction: If $add\ n_1\ n_2\ n_3$, then $add\ n_2\ n_1\ n_3$ (Commutative law of add).

(**Hint**: You can begin with proof of this lemma: If $n\ nat$, then $add\ n\ Z\ n$.)

Solution.

(a) **define** $add\ n_1\ n_2\ n_3$

$$\frac{}{add\ Z\ n\ n}AddZ \qquad \frac{add\ n_1\ n_2\ n_3}{add\ S(n_1)\ n_2\ S(n_3)}AddS$$

(b) **Proof:** We first prove the following lemmas.

Lemma 1. *If n nat, then $add\ n\ Z\ n$*

Proof: By induction on the derivation of $n\ nat$

1. **Case:** $\frac{}{Z\ nat}Z$
   Need to Prove: $add\ Z\ Z\ Z$

   $$(1)\ add\ Z\ Z\ Z \qquad (By\ AddZ\ and\ let\ n = Z)$$

2. **Case:** $\frac{n\ nat}{S(n)\ nat}S$
   Need to Prove: $add\ S(n)\ Z\ S(n)$

   $$(1)\ add\ n\ Z\ n \qquad (By\ I.H)$$
   $$(2)\ add\ S(n)\ Z\ S(n) \qquad (By\ (1)\ and\ AddS)$$

Lemma 2. *If $add\ n_1\ n_2\ n_3$, then $add\ n_1\ S(n_2)\ S(n_3)$*

Proof: By induction on the derivation of $add\ n_1\ n_2\ n_3$

1. **Case:** $\frac{}{add\ Z\ n\ n}AddZ$
   Need to Prove: If $add\ Z\ n\ n$, then $add\ Z\ S(n)\ S(n)$

   $$(1)\ add\ Z\ S(n)\ S(n) \qquad (By\ AddZ)$$

2. **Case:** $\frac{add\ n_1\ n_2\ n_3}{add\ S(n_1)\ n_2\ S(n_3)}AddS$
   Need to Prove: If $add\ S(n_1)\ n_2\ S(n_3)$, then $add\ S(n_1)\ S(n_2)\ S(S(n_3))$

   $$(1)\ add\ n_1\ n_2\ n_3 \qquad (By\ assumption)$$
   $$(2)\ add\ n_1\ S(n_2)\ S(n_3) \qquad (By\ (1)\ and\ I.H.)$$
   $$(3)\ add\ S(n_1)\ S(n_2)\ S(S(n_3)) \qquad (By\ (2)\ and\ AddS)$$

□

Then we prove the commutative law of add:

(Commutative law) If $add\ n_1\ n_2\ n_3$, then $add\ n_2\ n_1\ n_3$

Proof: By induction on the derivation of $add\ n_1\ n_2\ n_3$

1. **Case:** $\frac{}{add\ Z\ n\ n} AddZ$
   Need to Prove: If $add\ Z\ n\ n$, then $add\ n\ Z\ n$

$$(1)\ add\ n\ Z\ n \qquad\qquad (By\ lemma\ 1)$$

2. **Case:** $\frac{add\ n_1\ n_2\ n_3}{add\ S(n_1)\ n_2\ S(n_3)} AddS$
   Need to Prove: If $add\ S(n_1)\ n_2\ S(n_3)$, then $add\ n_2\ S(n_1)\ S(n_3)$

$$\begin{array}{ll} (1)\ add\ n_1\ n_2\ n_3 & (By\ assumption) \\ (2)\ add\ n_2\ n_1\ n_3 & (By\ (1)\ and\ I.H.) \\ (3)\ add\ n_2\ S(n_1)\ S(n_3) & (By\ (2)\ and\ lemma\ 2) \end{array}$$

□

Problem 4. (30 points) Recall the definition of natural numbers by n nat judgment taught in the lecture.

(a) (10 points) Give an inductive definition of the judgment form fib n_1 n_2, which indicates the n_1^{th} Fibonacci number is n_2.

(b) (10 points) Give an inductive definition of the judgment form fibsum n_1 n_2, which indicates the sum of the first n_1 Fibonacci numbers is n_2.

(c) (10 points) Prove by induction: If fibsum n m then fib succ(succ(n)) succ(m), that is

$$F_{n+2} = \sum_{i=1}^{n} F_i + 1.$$

Solution.

(a) **Define** $fib\ n_1\ n_2$

$$\frac{}{fib\ Z\ Z} fibZ \qquad \frac{}{fib\ S(Z)\ S(Z)} fibSZ$$

$$\frac{fib\ n_1\ n_2 \quad fib\ S(n_1)\ n_3 \quad add\ n_2\ n_3\ n_4}{fib\ S(S(n_1))\ n_4} fibS$$

You can reuse previously defined add here, or define it again. And you can assume fib numbers starts with 1 instead of 0, and the definition should be a little bit. That is also ok.

(b) **Define** $fibsum\ n_1\ n_2$

$$\frac{}{fibsum\ Z\ Z} fibsumZ \qquad \frac{fibsum\ n_1\ n_2 \quad fib\ S(n_1)\ n_3 \quad add\ n_2\ n_3\ n_4}{fibsum\ S(n_1)\ n_4} fibsumS$$

(c) **Proof:** By induction on the derivation of $fibsum\ n_1\ n_2$

1. **Case:** $\dfrac{}{fibsum\ Z\ Z}fibsumZ$
   Need to Prove: If $fibsum\ Z\ Z$, then $fib\ S(S(Z))\ S(Z)$

$$
\begin{aligned}
&(1)\ fib\ Z\ Z && (By\ fibZ)\\
&(2)\ fib\ S(Z)\ S(Z) && (By\ fibSZ)\\
&(3)\ add\ Z\ S(Z)\ S(Z) && (By\ AddZ)\\
&(4)\ fib\ S(S(Z))\ S(Z) && (By\ (1),(2),(3)\ and\ fibS)
\end{aligned}
$$

2. **Case:** $\dfrac{fibsum\ n_1\ n_2 \quad fib\ S(n_1)\ n_3 \quad add\ n_2\ n_3\ n_4}{fibsum\ S(n_1)\ n_4}\text{fibsumS}$
   Need to Prove: If $fibsum\ S(n_1)\ n_4$, then $fib\ S(S(S(n_1)))\ S(n_4)$

$$
\begin{aligned}
&(1)\ fibsum\ n_1\ n_2 && (By\ assumption)\\
&(2)\ fib\ S(S(n_1))\ S(n_2) && (By\ (1)\ and\ I.H.)\\
&(3)\ fib\ S(n_1)\ n_3 && (By\ assumption)\\
&(4)\ add\ n_2\ n_3\ n_4 && (By\ assumption)\\
&(5)\ add\ S(n_2)\ n_3\ S(n_4) && (By\ (4)\ and\ AddS)\\
&(6)\ add\ n_3\ S(n_2)\ S(n_4) && (By\ (5)\ and\ commutative\ law\ of\ add)\\
&(7)\ fib\ S(S(S(n_1)))\ S(n_4) && (By\ (2),(3),(6)\ and\ fibS)
\end{aligned}
$$

$\square$

Remark: You need to use **LaTeX** to write your homework and **convert it into .pdf** file. Please upload both **.tex and .pdf** files on **Canvas**.
File name format: HW_X_Name_StudentID.tex/HW_X_Name_StudentID.pdf