Solution 11 - Subtyping

∗ If there is any problem, please contact TA.
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Problem 1. (40 points) Remember in hw5, we extent tuples to records. Now we extend subtypes to records. Please give some subtyping rules for record type, then draw a derivation showing that $\{x : Nat, y : Nat, z : Nat\}$ is a subtype of $\{y : Nat\}$.

Solution.

$$e ::= ...|\{x_1 = e_1, ..., x_n = e_n\}|e.x$$
$$v ::= ...|\{x_1 = v_1, ..., x_n = v_n\}$$
$$t ::= ...|\{(x_1, t_1), ..., (x_n, t_n)\}$$

$$\frac{e_i \to e_i'}{\{x_1 = v_1, ..., x_{i-1} = v_{i-1}, x_i = e_i, ...\} \to \{x_1 = v_1, ..., x_{i-1} = v_{i-1}, x_i = e_i', ...\}} \quad (E - Record1)$$

$$\frac{e \to e'}{e.x \to e'.x} \quad (E - Label1)$$

$$\frac{}{\{x_1 = v_1, ..., x_n = v_n\}.x_i = v_i} \quad (E - Label2)$$

$$\frac{\Gamma \vdash e_i : t_i}{\Gamma \vdash \{x_1 = e_1, ..., x_n = e_n\} : \{(x_1, t_1), ..., (x_n, t_n)\}} \quad (T - Record)$$

$$\frac{\Gamma \vdash e : \{(x_1, t_1), ..., (x_n, t_n)\}}{\Gamma \vdash e.x_i : t_i} \quad (T - Label)$$

$$\frac{n \leq m}{\{l_i : T_i^{i \in 1...m}\} <= \{l_i : T_i^{1...n}\}} \quad (S - RecordWidth)$$

$$\frac{\forall 0 \leq i \leq n : S_i <= T_i}{\{l_i : S_i^{i \in 1...n}\} <= \{l_i : T_i^{1...n}\}} \quad (S - RecordDep)$$

$$\frac{0 \leq i \leq j \leq n}{\{l_0 : S_0, ..., l_i : S_i, ..., l_j : S_j, ..., l_n : S_n\} <= \{l_0 : S_0, ..., l_j : S_j, ..., l_i : S_i, ..., l_n : S_n\}} \quad (S - Exchange)$$

$$\frac{\overline{\{x{:}Nat, y{:}Nat, z{:}Nat\} <= \{y{:}Nat, x{:}Nat, z{:}Nat\}}\,(S - Exchange) \quad \overline{\{y{:}Nat, x{:}Nat, z{:}Nat\} <= \{y{:}Nat\}}\,(S - RecordWidth)}{\{x : Nat, y : Nat, z : Nat\} <= \{y; Nat\}}$$

□

Problem 2. (60 points) Prove Lemma [Inversion of the subtype relation]:
1. If $S <= T_1 \to T_2$, then S has the form $S_1 \to S_2$, with $T_1 <= S_1$ and $S_2 <= T_2$.
2. If $S <= \{l_i : T_i^{i \in l...n}\}$, then S has the form $\{k_j : S_j^{j \in l...m}\}$, with at least the labels $\{l_i^{i \in l...n}\}$ (i.e., $\{l_i^{i \in l...n}\} \subseteq \{k_j^{j \in l...m}\}$) and with $S_j <= T_i$ for each common label $l_i = k_j$.

Solution.

1. Prove by the derivation of $S <= T_1 \to T_2$

   (1.) Case $\frac{T_1 <= S_1 \quad S_2 <= T_2}{S_1 \to S_2 <= T_1 \to T_2}$ (S-Function)
   (1) $S = S_1 \to S_2$ (by assumption)
   (2) $T_1 <= S_1$ (by assumption)
   (3) $S_2 <= T_2$ (by assumption)

   (2.) Case $\frac{}{S <= S}$ (S-Reflexivity)
   (1) $S = T_1 \to T_2$ (by assumption)
   (2) $T_1 <= T_1$ (by S-Reflexivity)
   (3) $T_2 <= T_2$ (by S-Reflexivity)

   (3.) Case $\frac{S <= Q \quad Q <= T}{S <= T}$ (S-Transitivity)
   (1) $T = T_1 \to T_2$ (by assumption)
   (2) $Q <= T$ (by assumption)
   (3) $Q <= T_1 \to T_2$ (by (1) and (2))
   (4) $Q = Q_1 \to Q_2$ and $T_1 <= Q_1$ and $Q_2 <= T_2$ (by (3) and I.H.)
   (5) $S <= Q$ (by assumption)
   (6) $S <= Q_1 \to Q_2$ (by (4) and (5))
   (7) $S = S_1 \to S_2$ and $Q_1 <= S_1$ and $S_2 <= Q_2$ (by (6) and I.H.)
   (8) $T_1 <= Q_1 <= S_1$ and $S_2 <= Q_2 <= T_2$ (by (4), (7) and S-Transitivity)

2. Prove by the derivation of $S <= \{l_i : T_i^{i \in l \ldots n}\}$

   (1.) Case $\frac{}{S <= S}$ (S-Reflexivity)
   (1) $S <= \{l_i : T_i^{i \in l \ldots n}\}$ (by assumption)
   (2) $\{l_i^{i \in l \ldots n}\} \subseteq \{l_i^{i \in l \ldots n}\}$
   (3) $\forall l_i = l_i : T_i <= T_i$ (by S-Reflexivity)

   (2.) Case $\frac{S <= Q \quad Q <= T}{S <= T}$ (S-Transitivity)
   (1) $T = \{l_i : T_i^{i \in l \ldots n}\}$ (by assumption)
   (2) $Q <= T$ (by assumption)
   (3) $Q <= \{l_i : T_i^{i \in l \ldots n}\}$ (by (1) and (2))
   (4) $Q = \{w_p : Q_p^{p \in l \ldots r}\}$ and $\{l_i^{i \in 1 \ldots n}\} \subseteq \{w_p^{p \in 1 \ldots r}\}$ and $\forall l_i = w_p : Q_p <= T_i$ (by (3) and I.H.)
   (5) $S <= Q$ (by assumption)
   (6) $S <= \{w_p : Q_p^{p \in l \ldots r}\}$ (by (4) and (5))
   (7) $S = \{k_j : S_j^{j \in l \ldots m}\}$ and $\{w_p^{i \in 1 \ldots r}\} \subseteq \{k_j^{j \in 1 \ldots m}\}$ and $\forall w_p = k_j : S_j <= Q_p$ (by (6) and I.H.)
   (8) $\{l_i^{i \in 1 \ldots n}\} \subseteq \{w_p^{p \in 1 \ldots r}\} \subseteq \{k_j^{j \in 1 \ldots m}\}$ and $\forall l_i = k_j : S_j <= T_i$ (by (4), (7) and S-Transitivity)

(3.) Case $\frac{n \leq m}{\{l_i : T_i^{i \in 1 \dots m}\} <= \{l_i : T_i^{1 \dots n}\}}$(S-RecordWidth)

(1) $S = \{l_i : T_i^{i \in 1 \dots m}\}$ (by assumption)

(2) $n \leq m$ (by assumption)

(3) $\{l_i^{i \in l \dots n}\} \subseteq \{l_i^{i \in l \dots m}\}$ (by (2))

(4) $\forall 0 \leq i \leq n : T_i <= T_i$ (by S-Reflexivity)

(4.) Case $\frac{\forall 0 \leq i \leq n : S_i <= T_i}{\{l_i : S_i^{i \in 1 \dots n}\} <= \{l_i : T_i^{1 \dots n}\}}$(S-RecordDep)

(1) $S = \{l_i : S_i^{i \in 1 \dots n}\}$ (by assumption)

(2) $\{l_i^{i \in l \dots n}\} \subseteq \{l_i^{i \in l \dots n}\}$

(3) $\forall 0 \leq i \leq n : S_i <= T_i$ (by assumption)

(4) $\forall l_i = l_i : S_i <= T_i$ (by (3))

(5.) Case $\frac{0 \leq i \leq j \leq n}{\{l_0 : S_0, \dots, l_i : S_i, \dots, l_j : S_j, \dots, l_n : S_n\} <= \{l_0 : S_0, \dots, l_j : S_j, \dots, l_i : S_i, \dots, l_n : S_n\}}$(S-Exchange)

(1) $S = \{l_0 : S_0, ..., l_i : S_i, ..., l_j : S_j, ..., l_n : S_n\}$ (by assumption)

(2) $\{l_0, ..., l_i, ..., l_j, ..., l_n\} \subseteq \{l_0, ..., l_j, ..., l_i, ..., l_n\}$

(3) $\forall l_i = l_i : T_i <= T_i$ (by S-Reflexivity)

$\square$