Homework 10 - Inference2

∗ If there is any problem, please contact TA.
Name:__________ Student ID:__________ Email: ______________

Problem 1. Prove the Lemma: If $(S, q) \rightarrow (S', q')$ then:

- T is complete for (S, q) iff T is complete for (S', q')
- T is principal for (S, q) iff T is principal for (S', q')

Proof.

$$\frac{}{S| - \{\}} \quad \text{(S-empty)}$$

$$\frac{S(a) = S(b) \qquad S| = q}{S| = \{a = b\} \cup q} \quad \text{(S-equal)}$$

Lemma 1. *If $T(m) = T(n), T| = q$, then $T| = q[n/m]$*

Proof. Prove: By induction on the derivation of $S| = q$
case S-empty: obviously
case S-equal: If m=a or m=b .... else .... (Here we skip the proof steps)

And it's easy to prove the inversion lemma is also right, which is
If $T(m) = T(n), T| = q[n/m]$, then $T| = q$

□

Lemma 2. *If $T(a) = T(s), T <= S$, then $T <= [a = s] \circ S$*

Proof. Prove:Suppose $T = U \circ S$
Let $S' = U \circ [a = s] \circ S$, for all variables x
If $x \neq a$, T(x) = U(S(x)), S'(x)=U(S(x)) = T(x)
If $x = a$,
  if $a \in dom(S)$, S'(a)=U(S(a)) = T(a).
  if $a \notin dom(S)$, S'(a) = U([a=s](S(a))) = U([a=s](a)) = U(s)
    if $s \notin dom(S)$, T(a) = T(s) = U(S(s)) = U(s) = S'(a)
    if $s \in dom(S)$ T(a) = T(s) = U(S(s)), let $S' = U \circ [s = S(s)] \circ [a = s] \circ S$,
      S'(a) = U(S(s)) = T(a)
So $T = S'$. Because $S' <= [a = s] \circ S$, so $T <= [a = s] \circ S$

□

Now back to the problem.
By induction on the derivation of unification step

- **Case:** $\frac{}{(S,\{int=int\}\cup q)\rightarrow(S,q)}$**(u-int)**

  Need to prove: T is complete for $(S, \{int = int\} \cup q)$ iff T is complete for (S, q)

a) $\rightarrow$

$$\begin{aligned}
&(1)\ T\ is\ complete\ for\ (S, \{int = int\} \cup q) && (by\ assumption)\\
&(2)\ T <= S,\\
&\quad T| = \{int = int\} \cup q && (by\ (1))\\
&(3)\ T| = q && (by\ (2)\ and\ inversion\ of\ S-equal)\\
&(4)\ T\ is\ complete\ for\ (S, q) && (by\ (2)\ and\ (3))
\end{aligned}$$

b) $\leftarrow$

$$\begin{aligned}
&(1)\ T\ is\ complete\ for\ (S, q) && (by\ assumption)\\
&(2)\ T <= S,\\
&\quad T| = q && (by\ (1))\\
&(3)\ T(int) = T(int)\\
&(4)\ T| = \{int = int\} \cup q && (by\ (2), (3)\ and\ S-equal)\\
&(5)\ T\ is\ complete\ for\ (S, \{int = int\} \cup q) && (by\ (2)\ and\ (4))
\end{aligned}$$

Need to prove: T is principal for $(S, \{int = int\} \cup q)$ iff T is principal for (S', q')

a) $\rightarrow$

$$\begin{aligned}
&(1)\ T\ is\ principal\ for\ (S, \{int = int\} \cup q) && (by\ assumption)\\
&(2)\ T\ is\ complete\ for\ (S, \{int = int\} \cup q) && (by\ (1))\\
&(3)\ T\ is\ complete\ for\ (S', q') && (by\ (2))\\
&(4)\ For\ any\ complete\ solution\ T'\ for\ (S', q'),\\
&\quad T'\ is\ complete\ for\ (S, \{int = int\} \cup q)\\
&(5)\ T' <= T && (by\ (1))\\
&(6)\ T\ is\ principal\ for\ (S', q') && (by\ (3)\ and\ (5))
\end{aligned}$$

b) $\leftarrow$

$$\begin{aligned}
&(1)\ T\ is\ principal\ for\ (S', q') && (by\ assumption)\\
&(2)\ T\ is\ complete\ for\ (S', q') && (by\ (1))\\
&(3)\ T\ is\ complete\ for\ (S, \{int = int\} \cup q) && (by\ (2))\\
&(4)\ For\ any\ complete\ solution\ T'\ for\ (S, \{int = int\} \cup q),\\
&\quad T'\ is\ complete\ for\ (S', q')\\
&(5)\ T' <= T && (by\ (1))\\
&(6)\ T\ is\ principal\ for\ (S, \{int = int\} \cup q) && (by\ (3)\ and\ (5))
\end{aligned}$$

- **Case:** $\frac{}{(S, \{bool = boolt\} \cup q) \rightarrow (S, q)}$ **(u-bool)**
  Similar to u-int.

- **Case:** $\frac{}{(S,\{a=a\}\cup q)\rightarrow(S,q)}$**(u-eq)**

  Similar to u-int.

- **Case:** $\frac{}{(S,\{s_{11}\rightarrow s_{12}=s_{21}\rightarrow s_{22}\}\cup q)\rightarrow(S,\{s_{11}=s_{21},s_{12}=s_{22}\}\cup q)}$**(u-fun)**

  Need to prove: T is complete for $(S,\{s_{11}\rightarrow s_{12}=s_{21}\rightarrow s_{22}\}\cup q)$ iff T is complete for $(S,\{s_{11}=s_{21},s_{12}=s_{22}\}\cup q)$

  a) $\rightarrow$

  $$
  \begin{aligned}
  &(1)\ T\ is\ complete\ for\ (S,\{s_{11}\rightarrow s_{12}=s_{21}\rightarrow s_{22}\}\cup q) && (by\ assumption)\\
  &(2)\ T<=S,\\
  &\quad T|=\{s_{11}\rightarrow s_{12}=s_{21}\rightarrow s_{22}\}\cup q && (by\ (1))\\
  &(3)\ T(s_{11}\rightarrow s_{12})=T(s_{21}\rightarrow s_{22})\\
  &\ \rightarrow T(s_{11})\rightarrow T(s_{12})=T(s_{21})\rightarrow T(s_{22}) && (by\ (2)\ and\ inversion\ of\ S-equal)\\
  &(4)\ T(s_{11})=T(s_{21}),T(s_{12})=T(s_{22}) && (by\ (3))\\
  &(5)\ T|=\{s_{11}=s_{21},s_{12}=s_{22}\}\cup q && (by\ (4)\ and\ S-equal)\\
  &(6)\ T\ is\ complete\ for\ (S,\{s_{11}=s_{21},s_{12}=s_{22}\}\cup q) && (by\ (2)\ and\ (5))
  \end{aligned}
  $$

  b) $\leftarrow$

  $$
  \begin{aligned}
  &(1)\ T\ is\ complete\ for\ (S,\{s_{11}=s_{21},s_{12}=s_{22}\}\cup q) && (by\ assumption)\\
  &(2)\ T<=S,\\
  &\quad T|=\{s_{11}=s_{21},s_{12}=s_{22}\}\cup q && (by\ (1))\\
  &(3)\ T|=q\\
  &\quad T(s_{11})=T(s_{21})\\
  &\quad T(s_{12})=T(s_{22}) && (by\ (2)\ and\ inversion\ of\ S-equal)\\
  &(4)\ T(s_{11}\rightarrow s_{12})=T(s_{11})\rightarrow T(s_{12})\\
  &\ =T(s_{21})\rightarrow T(s_{22})=T(s_{21}\rightarrow s_{22}) && (by\ (3))\\
  &(5)\ T|=\{s_{11}\rightarrow s_{12}=s_{21}\rightarrow s_{22}\}\cup q && (by\ (3),(4)\ and\ S-equal))\\
  &(6)\ T\ is\ complete\ for\ (S,\{int=int\}\cup q) && (by\ (2)\ and\ (5))
  \end{aligned}
  $$

  Need to prove: T is principal for $(S,\{s_{11}\rightarrow s_{12}=s_{21}\rightarrow s_{22}\}\cup q)$ iff T is principal for $(S,\{s_{11}=s_{21},s_{12}=s_{22}\}\cup q)$

  Similar to u-int.

- **Case:** $\frac{}{(S,\{a=s\}\cup q)\rightarrow([a=s]\circ S,q[s/a])}$**(a not in FV(s))(u-var1)**

  Need to prove: T is complete for $(S,\{a=s\}\cup q)$ iff T is complete for $([a=s]\circ S,q[s/a])$

a) $\rightarrow$

$$
\begin{array}{ll}
(1)\ T\ is\ complete\ for\ (S, \{a = s\} \cup q) & (by\ assumption)\\
(2)\ T <= S, & \\
\quad T| = \{a = s\} \cup q & (by\ (1))\\
(3)\ T(a) = T(s) & \\
\quad T| = q & (by\ (2)\ and\ inversion\ of\ S-equal)\\
(4)\ T| = q[s/a] & (by\ (3)\ and\ lemma1)\\
(5)\ T <= [a = s] \circ S & (by\ (2), (3)\ and\ lemma2)\\
(6)\ T\ is\ complete\ for\ ([a = s] \circ S, q[s/a]) & (by\ (4)\ and\ (5))
\end{array}
$$

b) $\leftarrow$

$$
\begin{array}{ll}
(1)\ T\ is\ complete\ for\ ([a = s] \circ S, q[s/a]) & (by\ assumption)\\
(2)\ T <= [a = s] \circ S, & \\
\quad T| = q[s/a] & (by\ (1))\\
(3)\ T = U \circ [a = s] \circ S <= S & (by\ (2))\\
(4)\ a \notin dom(S), s \notin dom(S) & \\
(5)\ T(a) = T(s) & (by\ (4))\\
(6)\ T| = q & (by\ (2), (5)\ and\ inversion\ of\ lemma1)\\
(7)\ T| = \{a = s\} \cup q & (by\ (5), (6)\ and\ S-equal)\\
(7)\ T\ is\ complete\ for\ (S, \{a = s\} \cup q) & (by\ (3)\ and\ (6))
\end{array}
$$

Need to prove: T is principal for $(S, \{a = s\} \cup q)$ iff T is principal for $([a = s] \circ S, q[s/a])$

Similar to u-int.

- **Case:** $\frac{}{(S,\{s=a\}\cup q)\rightarrow([a=s]\circ S, q[s/a])}$**(a not in FV(s))(u-var2)**

  Similar to u-var2

∎

Problem 2. Given the following variant of untyped lambda calculus:

```
e::=
x (variables)
| c (constants)
| \x.e
| e1 e2
| e1 bop e2 (binary op)
| uop e (unary op)
| let x = e1 in e2
```

```
| if e1 then e2 else e3
| letfun f(x) = e1 in e2  (defining a recursive function f(x) for use in e2)
| {e1, e2}
| e.1
| e.2
| inl e
| inr e
| case e1 of inl x => e2 | inr x => e3
| nil
| e1 :: e2
| case e1 of nil => e2 | x1 :: x2 => e3
| (e)
```

(a) Inductively define the constraint generation judgement:

```
G |- u ==> e:t, q
```

(b) Give the detailed derivation of the following expressions and obtain the set of equations, then solve these equations to get the principle solution and give the universal polymorphic types:

```
letfun sum(l) = case l of nil => 0 | x1 :: x2 => x1 + sum(x2)
in sum(12::10::0::nil)
```

(a) *Solution.*

$$\frac{G(x) = t}{G \vdash x \Rightarrow x : t, \{\}} \qquad (CT - Var)$$

$$\frac{}{G \vdash c \Rightarrow c : int, \{\}}(c\ is\ integer) \qquad (CT - Int)$$

$$\frac{}{G \vdash c \Rightarrow c : bool, \{\}}(c\ is\ true\ or\ false) \qquad (CT - Bool)$$

$$\frac{G, x : t_1 \vdash t \Rightarrow t : t_2, q}{G, \lambda x : t_1.t : t_1 \rightarrow a,\ q \cup \{a = t_2\}} \qquad (CT - Abs)$$

$$\frac{G \vdash u_1 \Rightarrow e_1 : t_1, q_1 \quad G \vdash u_2 \Rightarrow e_2 : t_2, q_2}{G \vdash u_1\ u_2 \Rightarrow e_1\ e_2 : a,\ q_1 \cup q_2 \cup \{t_1 = t_2 \rightarrow a\}} \qquad (CT-App)$$

$$\frac{G \vdash u_1 \Rightarrow e_1 : t_1, q_1 \quad G \vdash u_2 \Rightarrow e_2 : t_2, q_2}{G \vdash u_1\ bop\ u_2 \Rightarrow e_1\ bop\ e_2 : a,\ q_1 \cup q_2 \cup \{t_1 = t_2 = a\}} \qquad (CT-Bop)$$

$$\frac{G \vdash u \Rightarrow e : t, q}{G \vdash uop\ u \Rightarrow uop\ e : a,\ q \cup \{t = a\}} \qquad (CT-Uop)$$

$$\frac{G \vdash u_1 \Rightarrow e_1 : t_1, q_1 \quad G \vdash u_2 \Rightarrow e_2 : t_2, q_2}{G \vdash let\ x = u_1\ in\ u_2 \Rightarrow let\ x = e_1\ in\ e_2 : a,\ q_1 \cup q_2 \cup \{t_2 = a\}} \qquad (CT-Let)$$

$$\frac{G \vdash u_1 \Rightarrow e_1 : t_1, q_1 \quad G \vdash u_2 \Rightarrow e_2 : t_2, q_2, \quad G \vdash u_2 \Rightarrow e_2 : t_2, q_2}{\begin{array}{c} G \vdash if\ u_1\ then\ u_2\ else\ u_3 \Rightarrow if\ e_1\ then\ e_2\ else\ e_3 : a, \\ q_1 \cup q_2 \cup q_3 \cup \{t_1 = bool, t_2 = t_3 = a\} \end{array}} \qquad (CT-If)$$

$$\frac{G, f : a \rightarrow b, x : a \vdash u_1 \Rightarrow e_1 : t_1, q_1\ G, f(x) : b \vdash u_2 \Rightarrow e_2 : t_2, q_2}{\begin{array}{c} G \vdash letfun\ f(x) = u_1\ in\ u_2 \Rightarrow letfun\ f(x : a) : b = e_1\ in\ e_2 : c, \\ q_1 \cup q_2 \cup q_3 \cup \{t_1 = b, t_2 = c\} \end{array}} \qquad (CT-Letfun)$$

$$\frac{G \vdash u_1 \Rightarrow e_1 : t_1, q_1 \quad G \vdash u_2 \Rightarrow e_2 : t_2, q_2}{G \vdash \{u_1, u_2\} \Rightarrow \{e_1, e_2\} : a * b,\ q_1 \cup q_2 \cup \{t_1 = a, t_2 = b\}} \qquad (CT-Pair)$$

$$\frac{G \vdash u \Rightarrow \{e_1, e_2\} : t_1 * t_2, q}{G \vdash u.1 \Rightarrow e_1 : a,\ q \cup \{t_1 = a\}} \qquad (CT-Proj1)$$

$$\frac{G \vdash u \Rightarrow \{e_1, e_2\} : t_1 * t_2, q}{G \vdash u.2 \Rightarrow e_2 : a,\ q \cup \{t_2 = a\}} \qquad (CT-Proj2)$$

$$\frac{G \vdash u \Rightarrow e : t, q}{G \vdash inl[a + b]\ u \Rightarrow inl[a + b]\ e : a + b,\ q \cup \{t = a\}} \qquad (CT-Inl)$$

$$\frac{G \vdash u \Rightarrow e : t, q}{G \vdash inr[a + b]\ u \Rightarrow inr[a + b]\ e : a + b,\ q \cup \{t = b\}} \qquad (CT-Inr)$$

$$\frac{G \vdash u \Rightarrow e : t_1 + t_2, q_1\ G, x_1 : t_1 \vdash u_1 \Rightarrow e_1 : t, q_2\ G, x_2 : t_2 \vdash u_2 \Rightarrow e_2 : t, q_3}{\begin{array}{c} G \vdash (case\ u\ of\ inl\ x_1 \Rightarrow u_1 | inr\ x_2 \Rightarrow u_2) \Rightarrow (case\ e : a + b\ of\ inl\ x_1 \Rightarrow e_1 | \\ inr\ x_2 \Rightarrow e_2) : c, q_1 \cup q_2 \cup q_3 \cup \{t_1 = a, t_2 = b, t = c\} \end{array}} \qquad (CT-Case)$$

$$\frac{}{G \vdash u \Rightarrow nil[t] : t\ list,} \qquad (CT-Ni)$$

$$\frac{G \vdash u_1 \Rightarrow e_1 : t, q_1 \quad G \vdash u_2 \Rightarrow e_2 : t\ list, q_2}{G \vdash u_1 :: u_2 \Rightarrow e_1 :: e_2 : a,\ q_1 \cup q_2 \cup \{a = t\ list\}} \qquad (CT-Cons)$$

$$\frac{G \vdash u \Rightarrow e : t_1\ list, q_1\ G \vdash u_1 \Rightarrow e_1 : t, q_2\ G, x_1 : t, x_2 : t_1\ list \vdash u_2 \Rightarrow e_2 : t, q_3}{\begin{array}{c} G \vdash (case\ u\ of\ nil[a] \Rightarrow u_1 | x_1 :: x_2 \Rightarrow u_2) \Rightarrow (case\ e : a\ list\ of\ nil[a] \Rightarrow e_1 | \\ x_1 :: x_2 \Rightarrow e_2) : b, q_1 \cup q_2 \cup q_3 \cup \{t_1 = a, t = b\} \end{array}} \qquad (CT-Casel)$$

$$\frac{G \vdash u \Rightarrow e : t, q}{G \vdash u \Rightarrow e : unit, q \cup \{t = unit\}} \qquad (CT-Unit)$$

$\square$

(b) `letfun sum(l) = case l of nil => 0 | x1 :: x2 => x1 + sum(x2)`
`in sum(12::10::0::nil)`

Solution.
derivation:

$(letfun\ sum(l:a) = case\ l\ of\ nil : b\ list => 0|x1 : c :: x2 : d => x1 + sum(x2) : d \to e$
$in\ sum(12 :: 10 :: 0 :: nil), \{\})$
(by CT-Bop)
$\to(letfun\ sum(l:a) = case\ l\ of\ nil : b\ list => 0|x1 : c :: x2 : d => (x1 + sum(x2)) : f$
$in\ sum(12 :: 10 :: 0 :: nil), \{d = a, c = e, f = c\}$
(by CT-Casel)
$\to(letfun\ sum(l:a) = (case\ l\ of\ nil\ list => 0|x1 :: x2 => x1 + sum(x2)) : g$
$in\ sum(12 :: 10 :: 0 :: nil) : e,$
$\{d = a, c = e, f = c, a = b\ list, c = int, f = int, int = g\}$
(by CT-Letfun)
$\to((letfun\ sum(l:a) = (case\ l\ of\ nil\ list => 0|x1 :: x2 => x1 + sum(x2)) : g$
$in\ sum(12 :: 10 :: 0 :: nil) : e)h,$
$\{d = a, c = e, f = c, a = b\ list, c = int, f = int, int = g, g = e, h = e\}$
(by CT-Cons)
$\to(letfun\ sum(l:a) = (case\ l\ of\ nil\ list => 0|x1 :: x2 => x1 + sum(x2)) : g$
$in\ sum(12 :: 10 :: 0 :: nil) : e,$
$\{d = a, c = e, f = c, a = b\ list, c = int, f = int, int = g, g = e, h = e, d = int\ list\}$

solve constraint set:

$(I, \{d = a, c = e, f = c, a = b\ list, c = int, f = int, int = g, g = e, h = e, d = int\ list\}$
$\to([d = a] \circ I, \{c = e, f = c, a = b\ list, c = int, f = int, int = g, g = e, h = e, a = int\ list\})$
$\to([c = e] \circ [d = a] \circ I, \{f = e, a = b\ list, e = int, f = int, int = g, g = e, h = e, a = int\ list\})$
$\to([f = e] \circ [c = e] \circ [d = a] \circ I, \{a = b\ list, e = int, int = g, g = e, h = e, a = int\ list\})$
$\to([a = b\ list] \circ [f = e] \circ [c = e] \circ [d = a] \circ I, \{e = int, int = g, g = e, h = e, b\ list = int\ list\})$
$\to([e = int] \circ [a = b\ list] \circ [f = e] \circ [c = e] \circ [d = a] \circ I, \{int = g, h = int, b\ list = int\ list\})$
$\to([g = int] \circ [e = int] \circ [a = b\ list] \circ [f = e] \circ [c = e] \circ [d = a] \circ I, \{h = int, b\ list = int\ list\})$
$\to([h = int] \circ [g = int] \circ [e = int] \circ [a = b\ list] \circ [f = e] \circ [c = e] \circ [d = a] \circ I, \{b\ list = int\ list\})$
$\to([b = int] \circ [h = int] \circ [g = int] \circ [e = int] \circ [a = b\ list] \circ [f = e] \circ [c = e] \circ [d = a] \circ I, \{\})$

pincipal solution: S(b)=S(c)=S(e)=S(f)=S(g)=S(h)int, S(d)=S(a)int list universal polymorphic types:

$letfun\ sum(l : int\ list) : int = case\ l\ of\ nil : int\ list => 0 | x1 : int :: x2 : int => x1 + sum(x2)$
$in\ sum(12 :: 10 :: 0 :: nil)$
$: int$

□

Problem 3. Show why type checking let expression using [t-LetPoly] is exponential in time and give an amortised linear implementation of let polymorphism instead.

Solution. Suppose the length of the input term e_0 is n. e_0 is a let expression like $let\ x = e_1\ in\ x\ x\ x\ x...$ and $e_1 = let\ x = e_2\ in\ x\ x\ x....$ The length of e_1 is $n/2$. Repeat this step so that e_1, e_2, e_3 have the same formulations as e_0. In this case the time complexity is $O(n/2) * O(n/4) * O(n/8).... = O(n^{\log n})$, which is exponential.

We can solve $let\ x = e_1\ in\ e_2$ in this way:

1. Once we get the principal type t_1 of e_1, we don't bind it with x in context Γ. We find all free variables in t_1. Suppose they are $x_1, .., x_n$. Now we bind x with a special type scheme $\forall x_1...x_n.t_1$.
2. We do typecheck for e_2. Each time we encounter an occurrence of x in e_2, we generate type variables $y_1, ...y_n$ and use them to instantiate $\forall x_1...x_n.t_1$, yielding $t_1[y_1/x_1, ..., y_n/x_n]$

□