# Gödel Number*

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## Outline







## General Remark

The set of the programs are countable.

More importantly, every program can be coded up effectively by a number in such a way that a unique program can be recovered from the number.





## Denumerability and Enumerability

A set $X$ is denumerable if there is a bijection $f : X \to \mathbb{N}$.

An enumeration of a set $X$ is a surjection $g : \mathbb{N} \to X$;
this is often represented by writing $\{x_0, x_1, x_2, \ldots\}$.
It is an enumeration *without repetitions* if $g$ is injective.

Let $X$ be a set of "finite objects".
Then $X$ is effectively denumerable if there is a bijection $f : X \to \mathbb{N}$ such that both $f$ and $f^{-1}$ are effectively computable functions.

## Effective Denumerability

**Fact.** $\mathbb{N} \times \mathbb{N}$ is effectively denumerable.

*Proof.* A bijection $\pi : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ is defined by

$$\begin{aligned} \pi(m,n) &\stackrel{\text{def}}{=} 2^m(2n+1) - 1, \\ \pi^{-1}(l) &\stackrel{\text{def}}{=} (\pi_1(l), \pi_2(l)), \end{aligned}$$

where

$$\begin{aligned} \pi_1(x) &\stackrel{\text{def}}{=} (x+1)_1, \\ \pi_2(x) &\stackrel{\text{def}}{=} ((x+1)/2^{\pi_1(x)} - 1)/2. \end{aligned}$$





## Effective Denumerability

**Fact.** $\mathbb{N}^+ \times \mathbb{N}^+ \times \mathbb{N}^+$ is effectively denumerable.

*Proof.* A bijection $\zeta : \mathbb{N}^+ \times \mathbb{N}^+ \times \mathbb{N}^+ \to \mathbb{N}$ is defined by

$$\begin{aligned} \zeta(m,n,q) &\stackrel{\text{def}}{=} \pi(\pi(m-1, n-1), q-1), \\ \zeta^{-1}(l) &\stackrel{\text{def}}{=} (\pi_1(\pi_1(l)) + 1, \pi_2(\pi_1(l)) + 1, \pi_2(l) + 1). \end{aligned}$$





## Effective Denumerability

**Fact.** $\bigcup_{k>0} \mathbb{N}^k$ is effectively denumerable.

*Proof.* A bijection $\tau : \bigcup_{k>0} \mathbb{N}^k \to \mathbb{N}$ is defined by

$$\begin{aligned} \tau(a_1,\ldots,a_k) \stackrel{\text{def}}{=}\ & 2^{a_1} + 2^{a_1+a_2+1} + 2^{a_1+a_2+a_3+2} + \ldots \\ & + 2^{a_1+a_2+a_3+\ldots,a_k+k-1} - 1. \end{aligned}$$

Now given $x$ we can find a unique expression of the form

$$2^{b_1} + 2^{b_2} + 2^{b_3} + \ldots + 2^{b_k}$$

that equals to $x+1$. It is then clear how to define $\tau^{-1}(x)$.





## Gödel Encoding

Let $\mathscr{I}$ be the set of all instructions.

Let $\mathscr{P}$ be the set of all programs.

The objects in $\mathscr{I}$, and $\mathscr{P}$ as well, are 'finite objects'.
They must be effectively denumerable.

## Gödel Encoding

**Theorem**. $\mathscr{I}$ is effectively denumerable.

*Proof*. The bijection $\beta : \mathscr{I} \to \mathbb{N}$ is defined as follows:

$$\begin{aligned} \beta(Z(n)) &= 4(n-1), \\ \beta(S(n)) &= 4(n-1)+1, \\ \beta(T(m,n)) &= 4\pi(m-1,n-1)+2, \\ \beta(J(m,n,q)) &= 4\zeta(m,n,q)+3. \end{aligned}$$

The converse $\beta^{-1}$ is easy.



Gödel Coding | Numbering Programs
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## Gödel Encoding

**Theorem**. $\mathscr{P}$ is effectively denumerable.

*Proof*. The bijection $\gamma : \mathscr{P} \to \mathbb{N}$ is defined as follows:

$$\gamma(P) = \tau(\beta(I_1), \ldots, \beta(I_s)),$$

assuming $P = I_1, \ldots, I_s$.

The converse $\gamma^{-1}$ is obvious.

We shall fix this particular coding function $\gamma$ throughout.



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## Gödel Encoding

The number $\gamma(P)$ is called the Gödel number of $P$.

$$\begin{aligned} P_n &= \text{the program with Gödel number } n \\ &= \gamma^{-1}(n) \end{aligned}$$



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## Gödel Encoding

Let $P$ be the program $T(1,3), S(4), Z(6)$.

$\beta(T(1,3)) = 18$, $\beta(S(4)) = 13$, $\beta(Z(6)) = 20$.

$\gamma(P) = 2^{18} + 2^{32} + 2^{53} - 1$.

## Gödel Encoding

Consider $P_{4127}$.

$4127 = 2^5 + 2^{12} - 1$.

$\beta(I_1) = 4 + 1$, $\beta(I_2) = 4\pi(1,0) + 2$.

So $P_{4127}$ is $S(2); T(2,1)$.





## Numbering Computable Functions

Suppose $a \in \mathbb{N}$ and $n \geq 1$.

$$\begin{aligned}
\phi_a^{(n)} &= \text{the } n \text{ ary function computed by } P_a \\
&= f_{P_a}^{(n)}, \\
W_a^{(n)} &= \text{the domain of } \phi_a^{(n)} = \{(x_1, \ldots, x_n) \mid P_a(x_1, \ldots, x_n) \downarrow\}, \\
E_a^{(n)} &= \text{the range of } \phi_a^{(n)}.
\end{aligned}$$

The super script $(n)$ is omitted when $n = 1$.





## Numbering Computable Functions

Let $a = 4127$. Then $P_{4127} = S(2); T(2,1)$.

$$\begin{aligned}
\phi_{4127}(x) &= 1, \\
W_{4127} &= \mathbb{N}, \\
E_{4127} &= \{1\}.
\end{aligned}$$

$$\begin{aligned}
\phi_{4127}^{(n)}(x_1, \ldots, x_n) &= x_2 + 1, \\
W_{4127}^n &= \mathbb{N}^n, \\
E_{4127}^n &= \mathbb{N}^+.
\end{aligned}$$





## Numbering Computable Functions

Suppose $f = \phi_a$. Then $a$ is an index for $f$.

There are an infinite number of indexes for $f$.

## Numbering Computable Functions

**Theorem**. $\mathscr{C}_n$ is denumerable.





## Proof

We use the enumeration $\phi_0^{(n)}, \phi_1^{(n)}, \phi_2^{(n)}, \cdots$ (with repetitions) to construct one without repetitions.

Let $\begin{cases} f(0) = 0; \\ f(m+1) = \mu z(\phi_z^{(n)} \neq \phi_{f(0)}^{(n)}, \cdots, \phi_{f(m)}^{(n)}), \end{cases}$

Then $\phi_{f(0)}^{(n)}, \phi_{f(1)}^{(n)}, \phi_{f(2)}^{(n)}, \cdots$ is an enumeration of $\mathscr{C}_n$ without repetitions.





## Corollary

**Corollary**: $\mathscr{C}$ is denumerable.

*Proof*: Since $\mathscr{C} = \bigcup_{n \geq 1} \mathscr{C}_n$, the corollary follows from the fact that a denumerable union of denumerable sets is denumerable.

Explicitly, for each $n$ let $f_n$ be the function to give an enumeration of $\mathscr{C}_n$ without repetitions. Let $\pi$ be the bijection $\mathbb{N} \times \mathbb{N} \to \mathbb{N}$. Define $\theta : \mathscr{C} \to \mathbb{N}$ by

$$\theta\left(\phi_{f_n(m)}^{(n)}\right) = \pi(m, n-1),$$

then $\theta$ is a bijection.



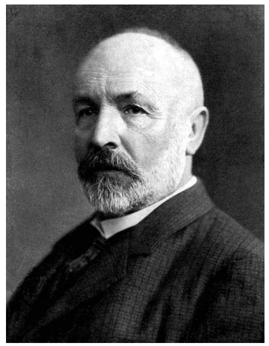


## Cantor's Diagonal Argument

In set theory, Cantor's diagonal argument, also called the diagonalisation argument, the diagonal slash argument or the diagonal method, was published in 1891 by Georg Cantor.

It was proposed as a mathematical proof for uncountable sets.

It demonstrates a powerful and general technique that has been used in a wide range of proofs.



**Georg Cantor**
1845-1918

## The Diagonal Method

**Theorem**. There is a total unary function that is not computable.

*Proof.* Suppose $\phi_0, \phi_1, \phi_2, \ldots$ is an enumeration of $\mathscr{C}_1$. Define

$$f(n) = \begin{cases} \phi_n(n) + 1, & \text{if } \phi_n(n) \text{ is defined,} \\ 0, & \text{if } \phi_n(n) \text{ is undefined.} \end{cases}$$

The function $f(n)$ is not computable.





## Example of uncomputable function

Consider again the construction of $f$ to construct a total uncomputable function. Complete details of the functions $\phi_0, \phi_1, \cdots$ can be represented by the following infinite table:

| | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| $\phi_0$ | $\phi_0(0)$ | $\phi_0(1)$ | $\phi_0(2)$ | $\phi_0(3)$ | $\ldots$ |
| $\phi_1$ | $\phi_1(0)$ | $\phi_1(1)$ | $\phi_1(2)$ | $\phi_1(3)$ | $\ldots$ |
| $\phi_2$ | $\phi_2(0)$ | $\phi_2(1)$ | $\phi_2(2)$ | $\phi_2(3)$ | $\ldots$ |
| $\phi_3$ | $\phi_3(0)$ | $\phi_3(1)$ | $\phi_3(2)$ | $\phi_3(3)$ | $\ldots$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | |





## Diagonal Method

We suppose that in this table the word 'undefined' is written whenever $\phi_n(m)$ is not defined.

The function $f$ was constructed by taking the diagonal entries on the table $\phi_0(0), \phi_1(1), \phi_2(2), \cdots$ and systematically changing them, obtaining $f(0), f(1), \cdots$ such that $f(n)$ differs from $\phi_n(n)$, for each $n$.

Note that there was considerable freedom in choosing the value of $f(n)$ (just differ from $\phi_n(n)$). Thus

$$g(n) = \begin{cases} \phi_n(n) + 27^n & \text{if } \phi_n(n) \text{ is defined,} \\ n^2 & \text{if } \phi_n(n) \text{ is undefined,} \end{cases}$$

is another non-computable total function.





## Cantor's Diagonal Method

Suppose that $\chi_0, \chi_1, \cdots$ is an enumeration of objects of a certain kind (functions or sets of natural numbers), then we can construct an object $\chi$ of the same kind that is different from every $\chi_n$, using the following motto:

'Make $\chi$ and $\chi_n$ differ at $n$.'

The interpretation of the phrase *differ at n* depends on the kind of object involved.

## Diagonal Construction on Sets

**Theorem**. The power set of $\mathbb{N}$ is not denumerable.

*Proof*: Contradiction. Suppose that $A_0, A_1, \cdots$ is an enumeration of subsets of $\mathbb{N}$. We can define a new set $B$ using the diagonal motto, by

$$n \in B \text{ if and only if } n \notin A_n.$$

Clearly, for each $n$, $B \neq A_n$.

Note that $B \in 2^{\mathbb{N}}$, but differs from any $A_i$ in the enumeration, so $2^{\mathbb{N}}$ is not a denumerable set.





## The s-m-n Theorem

Given a computable binary function $f(x,y)$ (not necessarily total), we get a unary computable function $f(a,y)$ by fixing a value $a$ for $x$.

We can use a unary computable function $g_a(y) \simeq f(a,y)$ to represent $f(a,y)$, then there is an index $e$ for $f(a,y)$.

$$f(a,y) \simeq \phi_e(y).$$

The S-m-n Theorem states that the index $e$ can be computed from $a$.





## The s-m-n Theorem, simple form

**Theorem**. Suppose that $f(x,y)$ is a computable function. There is a total computable function $k(x)$ such that

$$f(x,y) \simeq \phi_{k(x)}(y).$$





## The s-m-n Theorem

*Proof*. Let $F$ be a program that computes $f$. Consider the following program

$$\begin{array}{l} T(1,2) \\ Z(1) \\ \left.\begin{array}{l} S(1) \\ \vdots \\ S(1) \end{array}\right\} a \text{ times} \\ F \end{array}$$

The above program can be effectively constructed from $a$.

Let $k(a)$ be the Gödel number of the above program.
It can be effectively computed from the above program.

## Notation

The s-m-n theorem is also called Parametrization Theorem because it shows that an index for a computable function (such as $g_a$) can be found effectively from a parameter (such as $a$) on which it effectively depends.





## Examples

Let $f(x,y) = y^x$. Then $\phi_{k(x)}(y) = y^x$. For each fixed $n$, $k(n)$ is an index for $y^n$.

Let $f(x,y) = \begin{cases} y, & \text{if } y \text{ is a multiple of } x, \\ \text{undefined}, & \text{otherwise.} \end{cases}$.
Then $\phi_{k(n)}(y)$ is defined if and only if $y$ is a multiple of $n$.





## The s-m-n Theorem

**Theorem**. For $m, n$, there is a total computable $(m+1)$-function $s_n^m(\_, \mathbf{x})$ such that for all $e$ the following holds:

$$\phi_e^{m+n}(\mathbf{x}, \mathbf{y}) \simeq \phi^n_{s_n^m(e,\mathbf{x})}(\mathbf{y}).$$





## The s-m-n Theorem

*Proof.* Given $e, x_1, \ldots, x_m$, we can effectively construct the following program

$$\begin{array}{l} T(n, m+n) \\ \vdots \\ T(1, m+1) \\ Q(1, x_1) \\ \vdots \\ Q(m, x_m) \\ P_e \end{array}$$

where $Q(i, x_i)$ is the program $Z(i), \underbrace{S(i), \ldots, S(i)}_{x \text{ times}}$. □