De-anonymizing Social Networks with Overlapping Community Structure

Luoyi Fu[1], Jiapeng Zhang[2], Shuaiqi Wang[1], Xinyu Wu[2], Xinbing Wang[1,2], and Guihai Chen[1]

[1]Dept. of Computer Science, Shanghai Jiao Tong University, China.
[1]Email:{yiluofu, wangshuaiqi, xwang8} @sjtu.edu.cn, gchen@cs.sjtu.edu.cn.
[2]Dept. of Electrical Engineering, Shanghai Jiao Tong University, China.
[2]Email:{zhangjape, wuxinyu} @sjtu.edu.cn.

***Abstract*—The advent of social networks poses severe threats on user privacy as adversaries can de-anonymize users' identities by mapping them to correlated cross-domain networks. Without ground-truth mapping, prior literature proposes various cost functions in hope of measuring the quality of mappings. However, their cost functions, whose minimizers may remain algorithmically unknown, usually bring imponderable mapping errors when the true mapping cannot minimize these cost functions.**

We jointly tackle above concerns under a more practical social network model parameterized by *overlapping communities*, which, neglected by prior art, can serve as side information for de-anonymization. Regarding the unavailability of ground-truth mapping to adversaries, by virtue of the Minimum Mean Square Error (MMSE), our first contribution is a well-justified cost function minimizing the expected number of mismatched users over all possible true mappings. While proving the NP-hardness of minimizing MMSE, we validly transform it into the weighted-edge matching problem (WEMP), which, as disclosed theoretically, resolves the tension between optimality and complexity: (i) WEMP asymptotically returns a negligible mapping error in large network size under mild conditions facilitated by higher overlapping strength; (ii) WEMP can be algorithmically characterized via the convex-concave based de-anonymization algorithm (CBDA), effectively finding the optimum of WEMP. Extensive experiments further confirm the effectiveness of CBDA under overlapping communities: 90% users are re-identified averagely in a series of networks when communities overlap densely, and the re-identification ratio is enhanced about 70% compared to non-overlapping cases.

I. INTRODUCTION

With the mounting popularity of social networks, the privacy of users has been under great concern [1]–[3], as information of users in social networks is often released to public for wide usage in academy or advertisement [4]. Although users can be anonymized by removing personal identifiers such as names and family addresses, it is not sufficient for privacy protection since adversaries may re-identify these users by correlated side information.

Such user identification process in social networks resorting to auxiliary information is called *Social Network De-anonymization*. Initially proposed by Narayanan and Shimatikov [5], this fundamental issue has then gained increasing attention, leading to a large body of subsequent works [4], [6]–[11]. Particularly, this family of works embarked on de-anonymization under a common framework, as will also be the framework of interest in our setting. To elaborate, in the framework there is an underlying network G which characterizes the relationship among users. We can then observe two different networks G_1 and G_2 who contain the full node set of G and whose edges are sampled with probabilities s_1 and s_2 from G. The published network G_1 is anonymized by removing users' identities but preserves the structural information, while the users' identities in the auxiliary network G_2 are considered to be available for the public. *The aim of de-anonymization is to discover the correct mapping between V_1 and V_2, the node set of G_1 and G_2, which corresponds the same user in two networks, with the network structure as the only side information available to the adversaries.*

Regardless of the considerable efforts paid to the de-anonymization problem, there are still some deficiencies remaining unsolved. It can be accounted for from three aspects. (i) Analytically, despite a variety of existing works [6], [7] that proposed several cost functions in measuring the quality of mappings, the theoretical devise of those costs functions bases on the condition that the unique true mapping minimizes these cost functions. When this condition is not established, the mapping error will become unpredictable. (ii) Algorithmically, previous works [6], [7] failed to provide any algorithm to demonstrate that the optimal solution of proposed cost functions can indeed be effectively obtained. (iii) Experimentally, due to the destitution of real cross-domain datasets, state-of-the-art research [9], [10] simply evaluated the performance of proposed algorithms on synthetic datasets or real cross-domain networks formed by artificial sampling, falling short of reproducing the genuine social networks.

The above limitations motivate us to shed light on de-anonymization problem by jointly incorporating analytical, algorithmic and experimental aspects under the common framework noted earlier. As far as we know, the only work that shares the closest correlation with us belongs to Fu et al. [36], [37], who investigated this problem on social networks with non-overlapping communities and derived their cost function from the Maximum A Posterior (MAP) manner. However, we remark that the assumption of disjoint communities fails to reflect the real situation where a user belongs to multiple communities, as observed in massive real situations.

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For example, in social networks of scientific collaborators [11], actors and political blogospheres [12], users might belong to several research groups with different research topics, movies and political parties respectively. For this concern, by adopting the overlapping stochastic block model (OSBM), we allow the communities to overlap arbitrarily, which can well capture a majority of real social networks.

Meanwhile, though MAP enables adversaries to find the most possibly correct mapping, it relies heavily on a prerequisite, i.e., a hypothetically underlying true mapping between the given published and auxiliary networks. However, once the MAP estimation fails to exactly find this "true" mapping, the mapping error becomes unpredictable, with the probability that the estimation deviates largely from the real ground-truth. Therefore, we derive a new cost function based on Minimum Mean Square Error (MMSE), which minimizes the expected number of mismatched users by incorporating all the possible true mappings between the given published and auxiliary networks. Incorporating the two adjustments, from an average perspective, the result of our MMSE estimator is kept from significant deviation with any possible hypothetic true mapping.

Hereinafter we unfold our main contributions in analytical, algorithmic and experimental aspects respectively as follows:

1. Analytically, we are the first to derive cost function based on MMSE, which justifiably ensures the minimum expected mapping error between our estimation and the ground-truth mapping. Then we demonstrate the NP-hardness of solving MMSE, whose intractability stems mainly from the calculation of all $n!$ possible mappings (n is the total number of users). To cope with the hardness, we simplify MMSE by transforming it into a weighted-edge matching problem (WEMP), with mapping error negatively related to weights.

2. Algorithmically, in terms of solving WEMP, we theoretically reveal that WEMP alleviates the tension between optimality and complexity: Solving WEMP ensures optimality since its optimum, in large network size, negligibly deviates from the ground-truth mapping under mild conditions where on average a user belongs to asymptotically non-constant communities. Meanwhile, it reduces complexity since perfectly deriving its optimum only entails a convex-concave based de-anonymization algorithm (CBDA) with polynomial time. The proposed CBDA serves as one of the very few attempts to address the algorithmic characterization, that has long remained open, of de-anonymization without pre-identification.

3. Experimentally, we validate our theoretical findings that minimizing WEMP indeed incurs negligible mapping error in large social networks based on real datasets. Interestingly, we also observe significant benefits that community overlapping effect brings to the performance of CBDA: (i) in notable true cross-domain co-author networks with dense overlapping communities, CBDA can correctly re-identify 90% nodes on average; (ii) the overlapping communities bring about an enhancement of around 70% re-identification ratio compared with non-overlapping cases.

Unlike de-anonymization with pre-identified seed nodes, to which a family of work pays endeavor, no prior knowledge of such seeds complicates this problem, thus leaving many aspects largely unexplored. Meanwhile, theoretical results on such seedless cases in prior art is short of experimental verification. Our work is, as far as we are concerned, the initial devotion to theoretically dissecting seedless cases with overlapping communities, under real cross-domain networks with more than 2000 nodes. With novel exploitations of structural information, future design of more efficient mechanisms will be expected to further dilute the limitation of network size.

II. Related Works

Social network de-anonymization problem, initially proposed by Narayanan and Shmatikov [5], has been in the dimelight for decades. A major classification of this problem depends on whether the anonymized network is supported with side information, such as seed nodes [16]–[18] or community structures [20], [36], [37], etc. Mostly, it is considered to be difficult for the adversaries to obtain side information before the de-anonymization process due to their limited access to user profiles. In this case, a major methodology is to propose cost functions and realize an estimation of the correct mapping between two networks by optimizing these cost functions. Pedarsani and Grossglauer [6] first study this problem under Erdös-Rényi networks and set their cost function as the number of mismatched edges. This cost function still makes sense in a more general situation where the nodes in two networks are partially overlapping [7]. Further, Cullina and Kiyavash [8] investigated the information-theoretic threshold for exact identification in [6]. However, this cost function ensures optimality only when the true mapping has the smallest number of mismatched edges, while the mapping errors are imponderable if the structures of the observed networks do not suit for the high requirements raised by these works. To improve the robustness of the optimal solution, another cost function based on Maximum A Posterior (MAP) has also been justified by state-of-the-art literature. The validity of MAP is theoretically proved in [4] and approximation algorithms [36], [37] have been proposed to solve this problem.

Still in some cases, the side information is considered to be accessible to the adversaries. For instance, in the seeded de-anonymization, a handful of nodes will be pre-identified as seeds. As the pioneering work [5] has done, a well-known approach to solve this problem is to design algorithms based on *bootstrap percolation*, which means that the re-identification process starts from the seed nodes and identifies the neighbors of all identified nodes iteratively until all the nodes are de-anonymized. Yartseva et al. [16], Kazemi et al. [17] and Bringmann et al. [18] study the seeded case under the Erdös-Rényi network model, among which the de-anonymization process can be achieved as fast as a quasilinear ($O(n \log n)$) computational complexity. Meanwhile, some other works such as [19] also shed light on the preferential attachment model and propose correspondingly efficient algorithms. However, this series of methods may bring cumulative errors once a node is mismatched at early stage.

Another concern for the side information is the communities. A large volume of literature [5]–[8], [16]–[19], either in seeded or seedless situations, studies this problem considering only the topological structure of two networks, i.e., the edge

sets in two networks. However, the clustering effect exists in real social networks, which has not been seriously considered in these works. On the one hand, to incorporate the clustering effect, Nilizadeh et al. [20] detect the community structure before solving the seeded de-anonymization problem and find the superiority of clustering with respect to the noise, the number of seeds and the network size. Meanwhile, clustering is claimed to bring double-edged impact [21], which may dramatically reduce the required seed nodes but make the algorithm more fragile to errors. On the other hand, the side information of communities is also proved to make for higher accuracy of the algorithms intended in seedless case [36], [37]. However, the communities in real-world social networks are more likely to be overlapped with one another [22], [23], which, as far as we know, has not been concerned by existing works in the de-anonymization area. In this paper, we focus on the omnipresent overlapping communities in social networks, and de-anonymize these networks with a universal cost function based on the MMSE estimation.

III. Models and Definitions

In this section, we will introduce the fundamental model and the social network de-anonymization problem along with some related definitions. After that, we list some basic notations frequently used in our later analysis.

A. Social Network Models

The social network model considered in this paper is composed of three parts, i.e., the underlying network G, the published network G_1 and the auxiliary network G_2. G_1 and G_2 can be viewed as the incomplete observations of G, which represents the underneath relationship among all users. For instance, in reality G may characterize the true underlying relationship among a group of people, while G_1 might represent the online network in Facebook of this group of people and G_2 might represent the communication records in the cell phones of them, both of which are observable.

1) Underlying Social Network: Let $G = (V, E, \mathbf{U})$, where V is the node set, E is the edge set and $\mathbf{U}$ is the adjacent matrix. We regard G as undirected with $|V| = n$ nodes. To reflect the property of overlapping communities, we suppose G is generated based on the overlapping stochastic block model (OSBM) [12], whose idea can be interpreted as follows:

Suppose there are Q communities and each community q contains a subset of nodes. For a generic node i, we introduce a latent Q-dimensional column vector $\boldsymbol{C_i}$, in which all elements are independent boolean variables $C_{iq} \in \{0, 1\}$, with C_{iq} being the q_{th} row in $\boldsymbol{C_i}$. $C_{iq} = 1$ means that node i is in community q and $C_{iq} = 0$ otherwise. We denote p_q to be the probability of any node in G belonging to community q, thus we have: $Pr(\boldsymbol{C_i} = \{C_{i1}, C_{i2}, \cdots, C_{iQ}\}^T) = \prod_{q=1}^{Q} (p_q)^{C_{iq}} (1 - p_q)^{1-C_{iq}}$. We call $\boldsymbol{C_i}$ the *community representation* of node i, since $\boldsymbol{C_i}$ shows to which communities node i belongs exactly.

OSBM can measure the overlapping property of communities, allowing one node to belong to multiple communities, which is more realistic than the traditional SBM model. Further, the SBM is actually a special case of the OSBM if requiring $|\boldsymbol{C_i}| = 1$ for all i.

In OSBM, the probability of edge existence between nodes i and j in G relies on $\boldsymbol{C_i}$ and $\boldsymbol{C_j}$, where the communities can be detected with varied methods such as [23]–[25]. For instance, we can simply divide the nodes into different communities by k-cliques, where the nodes belong to one community if every of them connects to at least k nodes in this community.

Given $\boldsymbol{C_i}, i = 1, \ldots, Q$, we denote $V_{\boldsymbol{C_i}}$ as the node set of $\boldsymbol{C_i}$ and $E_{\boldsymbol{C_i}\boldsymbol{C_j}}$ as the set of edges between nodes in $\boldsymbol{C_i}$ and nodes in $\boldsymbol{C_j}$. Hence, the edge existence probability $p_{\boldsymbol{C_i}\boldsymbol{C_j}}$ for $1 \le i, j \le n$ can be estimated as follows:

$$p_{\boldsymbol{C_i}\boldsymbol{C_j}} = \begin{cases} \dfrac{|E_{\boldsymbol{C_i}\boldsymbol{C_j}}|}{|V_{\boldsymbol{C_i}}||V_{\boldsymbol{C_j}}|}, & \text{if } i \neq j, \\ \dfrac{|E_{\boldsymbol{C_i}\boldsymbol{C_i}}|}{\binom{|V_{\boldsymbol{C_i}}|}{2}}, & \text{if } i = j. \end{cases}$$

2) Published Network and Auxiliary Network: The two observed networks are sampled from the underlying network G, which have same node set with G. In details, we denote $G_1(V_1, E_1, \mathbf{A})$ as the published network, whose nodes may be labeled from 1 to n in a casual order and edges are independently sampled from G with probability s_1. In contrast, an auxiliary network, denoted by $G_2(V_2, E_2, \mathbf{B})$, has identical node labels with the underlying graph G, and the edges are independently sampled from G with probability s_2. Mathematically, we have $|V| = |V_k| = n$, $Pr\left((i, j) \in E_k | (i, j) \in E\right) = s_k$ and $Pr\left((i, j) \in E_k | (i, j) \notin E\right) = 0$ for $k \in \{1, 2\}$. $\mathbf{A}$ and $\mathbf{B}$, respectively, represent the adjacency matrices of G_1 and G_2. In correspondence to real situations, G_1 characterizes the anonymized network where users' identities are removed for privacy concern. On the contrary, G_2 characterizes a sanitized network where users' identities are available for adversaries.

Adversaries can leverage G_2 to identify nodes in G_1 based on the edge relationship and community information: (i) For edge relationship, adversaries can harness the *degree similarity* that a node of high degree in G_1 should be inclined to match a node of high degree in G_2; (ii) For community information, adversaries can exploit the *community representation similarity* that nodes in G_1 and G_2 with the same community representation should be matched with higher probability.

Furthermore, we should clarify that we render each node pair (i, j) a weight w_{ij}, which, quantified in Section III-B, is the cost of mistakenly matching the node pair (i, j) and is contingent on $p_{\boldsymbol{C_i}\boldsymbol{C_j}}$, s_1 and s_2. As we will show in Section IV-B, w_{ij} is negatively proportional to the number of communities nodes i and j co-exist in, evincing the cost reduction arose from higher *overlapping strength* of communities.

In fact, G, G_1 and G_2 are all random variables under the provided settings. We directly use G, G_1, G_2 as notations for the realizations of these random variables when there is no loss of clearance. Moreover, we set $\boldsymbol{\theta} = \{\{p_{\boldsymbol{C_i}\boldsymbol{C_j}} | 1 \le i, j \le n\}, s_1, s_2\}$ as the parameter set incorporating all pre-defined parameters in the model.

B. Social Network De-anonymization

The goal of social network de-anonymization problem is to find a mapping $\pi : V_1 \mapsto V_2$, which finds a corresponding

node in G_2 for every node in G_1. $\pi(i) = j$ means that node i in network G_1 is mapped to node j in network G_2 and $\pi(i) = 0$ when no nodes in network G_2 can be mapped with node i in G_1 (this may happen if $|V_1| \neq |V_2|$). We can equivalently express this mapping by forming a permutation matrix $\mathbf{\Pi} \in \{0,1\}^{n\times n}$, where $\mathbf{\Pi}(i,j) = 1$ if $\pi(i) = j$ and $\mathbf{\Pi}(i,j) = 0$ otherwise (If $|V_1| \neq |V_2|$, $\mathbf{\Pi}$ will be a non-square matrix which could be still in accordance with our analysis and algorithm design). By denoting $\mathbf{\Pi_0}(\pi_0)$ as the true permutation matrix (mapping) between G_1 and G_2, we can formally define the social network de-anonymization problem in Definition 1 along with an illustrative instance in Fig. 1. But in the process of achieving this goal, we have no prior knowledge of $\mathbf{\Pi_0}$ and no access to the underlying graph G, which triggers that our estimated permutation, $\hat{\mathbf{\Pi}}$, may deviate from the ground-truth $\mathbf{\Pi_0}$. To quantify this difference, we introduce a metric called "node mapping error (NME)" as Definition 2.

Definition 1. ***(Social Network De-anonymization Problem)*** *Given the published network $G_1(V_1, E_1, \mathbf{A})$, the auxiliary network $G_2(V_2, E_2, \mathbf{B})$ and the parameter set $\boldsymbol{\theta} = \{\{p_{\boldsymbol{C_i}\boldsymbol{C_j}}|1 \leq i,j \leq n\}, s_1, s_2\}$, social network de-anonymization problem aims to construct the true mapping $\pi_0 : V_1 \mapsto V_2$ that correctly maps all nodes in G_1 to their correspondences in G_2.*

Definition 2. ***(Node Mapping Error, NME)*** *Given the estimated $\hat{\mathbf{\Pi}}$ and ground-truth $\mathbf{\Pi_0}$, the node mapping error (NME) between $\hat{\mathbf{\Pi}}$ and $\mathbf{\Pi_0}$ is defined as $d(\hat{\mathbf{\Pi}}, \mathbf{\Pi_0}) = \frac{1}{2}||\hat{\mathbf{\Pi}} - \mathbf{\Pi_0}||_F^2$.*

Obviously $d(\hat{\mathbf{\Pi}}, \mathbf{\Pi_0})$ equals to 0 if and only if two permutations are identical. If k nodes are mapped mistakenly, the NME equals to k, which means that NME is well-defined. Thus we can transform the goal of de-anonymization to minimize NME.

Moreover, since adversaries are uncertain about the true mapping between the given G_1 and G_2, $\mathbf{\Pi_0}$ can be viewed as a random variable whose probability distribution is conditioned on G_1 and G_2 in adversaries' perspectives. Naturally adversaries prefer an estimation of $\mathbf{\Pi_0}$ keeping from severe NME on average. To this end, we consider selecting $\hat{\mathbf{\Pi}}$ in the light of "Minimum Mean Square Error (MMSE)" criterion, which, formally presented in Definition 3, is the minimizer of the expected NME in the form of mean square.

Definition 3. ***(The MMSE Estimator)*** *Given G_1, G_2 and $\boldsymbol{\theta}$, the MMSE estimator is an estimation of $\mathbf{\Pi_0}$ minimizing the number of mistakenly matched nodes in expectation, which is*

$$\begin{aligned}\hat{\mathbf{\Pi}} &= \arg\min_{\mathbf{\Pi}\in\mathbf{\Pi^n}} \mathbf{E}_{\mathbf{\Pi_0}}\{d(\mathbf{\Pi}, \mathbf{\Pi_0})\} \\ &= \arg\min_{\mathbf{\Pi}\in\mathbf{\Pi^n}} \sum_{\mathbf{\Pi_0}\in\mathbf{\Pi^n}} ||\mathbf{\Pi} - \mathbf{\Pi_0}||_F^2 Pr(\mathbf{\Pi_0}|G_1, G_2, \boldsymbol{\theta}),\end{aligned} \tag{1}$$

where $\mathbf{\Pi^n}$ is the set of $n \times n$ permutation matrices.

Remark: Recall that prior effort [4] has leveraged Maximum A Posterior (MAP), which provides the solution with the highest probability being exactly identical to the true permutation. MMSE and MAP characterize different aspects of minimizing NME. As far as we know, no previous work has learned de-anonymization under MMSE, which, however, is also of great significance as MAP in reducing NME.

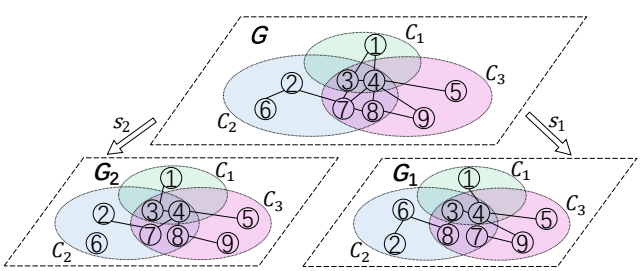


Fig. 1: An example of G, G_1 and G_2. The edges of $G_{1(2)}$ are sampled independently from G with probability $s_{1(2)}$. C_1, C_2, C_3 denote 3 different communities in OSBM. The true mapping $\pi_0 = \{(1,1),(2,6),(3,3),(4,4),(5,5),(6,2),(7,8),(8,7),(9,9)\}$.

TABLE I: **Notions and Definitions**

Notation	Definition
G	Underlying social network
G_1, G_2	Published and auxiliary networks
V, V_1, V_2	Vertex sets of graphs G, G_1 and G_2
E, E_1, E_2	Edge sets of graphs G, G_1, G_2
s_1, s_2	Edge sampling probabilities of graphs G_1, G_2
n	Total number of nodes
w_{ij}	The weight of node pair (i,j)
$\boldsymbol{C_i}$	Community representation of node i
$p_{\boldsymbol{C_i}\boldsymbol{C_j}}$	Edge existence probability between nodes i and j in G
$\boldsymbol{\theta}$	Parameter set, including $p_{\boldsymbol{C_i}\boldsymbol{C_j}}$ for $1 \leq i,j \leq n$, and s_1, s_2
$\mathbf{W}$	The weight matrix whose elements are determined by $\boldsymbol{\theta}$
$\mathbf{U}, \mathbf{A}, \mathbf{B}$	Adjacency matrices of G, G_1, G_2
$\mathbf{\Pi_0}(\pi_0)$	The true permutation matrix (true mapping) between V_1 and V_2
$\mathbf{\Pi}(\pi)$	A permutation matrix (A mapping) between V_1 and V_2
$\hat{\mathbf{\Pi}}(\hat{\pi})$	The solution of MMSE estimator (the corresponding mapping)
$\tilde{\mathbf{\Pi}}(\tilde{\pi})$	The optimal solution of WEMP (the corresponding mapping)
$\mathbf{\Pi^n}$	The set of all $n \times n$ permutation matrices
$\mathcal{G}_{\mathbf{\Pi}}$	the set of all feasible underlying networks w.r.t. G_1, G_2 and Π

C. Preliminary Notations

The main notations in our work are summarized in Table 1.

We also introduce some terms that could be useful in the derivations of Section IV as follows.

(i) **Expectation Over Matrix.** Given a random matrix variable $\mathbf{A}$ and a function $f(\mathbf{A})$, the expectation of $f(\mathbf{A})$ on matrix $\mathbf{A}$ is denoted as $\mathbf{E_A}(f(\mathbf{A})) = \sum_{\mathbf{A}} f(\mathbf{A})p(\mathbf{A})$.

(ii) **Frobenius Norm.** Given an $m \times n$ matrix $\mathbf{X}$, the Frobenius norm of $\mathbf{X}$ is $||\mathbf{X}||_F = \sqrt{\sum_{i=1}^{m}\sum_{j=1}^{n}(\mathbf{X}_{ij}^2)}$, where $\mathbf{X}_{ij}$ is the element at the i_{th} row and j_{th} column of $\mathbf{X}$.

(iii) **Hadamard Product.** Given two $n \times n$ matrices $\mathbf{Y}$ and $\mathbf{Z}$, The Hadamard Product between $\mathbf{Y}$ and $\mathbf{Z}$ is defined as $\forall i,j \in \{1,2,...,n\}, (\mathbf{Y} \circ \mathbf{Z})_{ij} = \mathbf{Y}_{ij}\mathbf{Z}_{ij}$. In operation, the matrix multiplication is prior to the Hadamard Product.

IV. Analytical Aspect of De-anonymization Problem

In this section, we start to provide analysis of the social network de-anonymization problem that we have defined earlier. First, we briefly prove that this problem is NP-hard. To facilitate the problem analysis, we then transform it into a matrix form and give an approximation to the MMSE estimator and verify it under the expectation of different possible network structures. Furthermore, we validate this approximation by proving that the approximation ratio is no less than one half in an average sense.

A. NP-hardness of Solving the MMSE Estimator

Since we have proposed the MMSE estimator, we are interested in whether there exists a polynomial-time algorithm that can solve the MMSE problem. However, as we will prove in the sequel, this problem is NP-hard, meaning that no polynomial time (pseudo-polynomial time) approximation algorithm exists for solving the MMSE estimator.

Proposition 1. *Solving the MMSE estimator is an NP-hard problem. There is no polynomial time or pseudo-polynomial time approximation algorithm for this problem with any multiplicative approximation guarantee unless P=NP.*

Proof. To begin with, we first introduce the linear assignment problem (LAP) as follows:

$$\textbf{LAP:} \ \max_{\Pi \in \Pi^n} \operatorname{tr}(C\Pi^T). \tag{2}$$

Here C is a cost matrix. Since this problem in (2) is known to be NP-hard [32], [33], we can then reduce it to our MMSE problem to prove the NP-hardness of the MMSE problem.

Given $\|\Pi - \Pi_0\|_F^2 = \|\Pi\|_F^2 + \|\Pi_0\|_F^2 - 2\text{tr}(\Pi_0\Pi^T) = 2n - 2\text{tr}(\Pi_0\Pi^T)$, Eqn. (1) can be reshaped as

$$\begin{aligned}\hat{\mathbf{\Pi}} &= \arg\min_{\mathbf{\Pi}\in\mathbf{\Pi^n}} \sum_{\mathbf{\Pi_0}\in\mathbf{\Pi^n}} (2n - 2\text{tr}(\mathbf{\Pi_0}\mathbf{\Pi}^T)) Pr(\mathbf{\Pi_0}|G_1, G_2, \boldsymbol{\theta}) \\ &= \arg\max_{\mathbf{\Pi}\in\mathbf{\Pi^n}} \text{tr}\left(\left(\sum_{\mathbf{\Pi_0}\in\mathbf{\Pi^n}} \mathbf{\Pi_0} Pr(\mathbf{\Pi_0}|G_1, G_2, \boldsymbol{\theta})\right)\mathbf{\Pi}^T\right).\end{aligned} \tag{3}$$

We denote $C = \sum_{\mathbf{\Pi_0}\in\mathbf{\Pi^n}} \mathbf{\Pi_0} Pr(\mathbf{\Pi_0}|G_1, G_2, \boldsymbol{\theta})$. Regardless of the $\Omega(n!)$ complexity to compute the value of matrix C, the problem in Eqn. (3) is an LAP which has been proved to be an NP-hard problem. □

The NP-hardness of MMSE estimator shows the impossibility to pursue an exact algorithm or any approximation algorithm with multiplicative guarantee. Thus we need to simplify this problem by conducting reasonable approximation to make it possible to solve this problem, with certain tolerance of mapping error. In the following, we transform it into a matrix form and propose one way to approximate this problem, the analysis of which will indicate that the error arose by this approximation can be bounded.

B. Transformation of MMSE Estimator

As can be seen from the definition of MMSE estimator (Eqn. (1) in Section III-B), the posterior probability $Pr(\mathbf{\Pi_0}|G_1, G_2, \boldsymbol{\theta})$ still needs to be expressed more explicitly. Inspired by the derivation in [4], we have the following theorem about the transformation of MMSE estimator.

Theorem 1. *Given the published graph G_1, the auxiliary graph G_2 and the parameter set $\boldsymbol{\theta}$, the MMSE estimator can be equivalently transformed into*

$$\hat{\mathbf{\Pi}} = \arg\max_{\mathbf{\Pi}\in\mathbf{\Pi^n}} \sum_{\mathbf{\Pi_0}\in\mathbf{\Pi^n}} ||\mathbf{\Pi} - \mathbf{\Pi_0}||_F^2 ||\mathbf{W} \circ (\mathbf{\Pi_0}\mathbf{A} - \mathbf{B}\mathbf{\Pi_0})||_F^2 \tag{4}$$

where in the metric $\mathbf{W}$, $W(i,j) = W(j,i) = \sqrt{w_{ij}}$, $w_{ij} = \log\left(\frac{1-p_{\boldsymbol{C_i}\boldsymbol{C_j}}(s_1+s_2-s_1s_2)}{p_{\boldsymbol{C_i}\boldsymbol{C_j}}(1-s_1)(1-s_2)}\right)$ is weight between nodes i and j, and "$\circ$" denotes the Hadamard product.

We shall provide here a sketch of the proof of Theorems 1. The complete proof, including all mathematical details, can be found in Appendix A.

Proof. Since the underlying network G is unknown in the de-anonymizing process, we first figure out the set of all feasible underlying networks. Given the two observed networks G_1 and G_2 as well as the true permutation matrix $\mathbf{\Pi_0}$, we can reshape the simplest underlying network G which is formed by the union of G_1 and G_2. Further, an arbitrary number of new edges can be added into network G to form another potential underlying network. In order for the convenience of illustration, we denote $\mathcal{G}_{\mathbf{\Pi}}$ as the set of all possible realizations of the underlying network which can be sampled to form the observed networks G_1, G_2 given the mapping matrix $\mathbf{\Pi}$. Then the MMSE estimator in Eqn. (4) can be transformed into

$$\hat{\mathbf{\Pi}} = \arg\min_{\mathbf{\Pi}\in\mathbf{\Pi^n}} \sum_{\mathbf{\Pi_0}\in\mathbf{\Pi^n}} ||\mathbf{\Pi} - \mathbf{\Pi_0}||_F^2 \sum_{G\in\mathcal{G}_{\mathbf{\Pi}}} Pr(G, \mathbf{\Pi_0}|G_1, G_2, \boldsymbol{\theta}). \tag{5}$$

Focusing on $Pr(G, \mathbf{\Pi_0}|G_1, G_2, \boldsymbol{\theta})$ in Eqn. (5), we derive that

$$Pr(G, \mathbf{\Pi_0}|G_1, G_2, \boldsymbol{\theta}) \ \sim \ Pr(G)Pr(G_1|G)Pr(G_2|G, \mathbf{\Pi_0}) \tag{6}$$

by Bayes formula as well as the independency of the sampling process of G_1 and G_2. What's more, $a \sim b$ means that a and b are positively correlated and their difference appears only in the parameters unrelated to $\mathbf{\Pi_0}$, which will not change the value of $\arg\min$ in Eqn. (5).

To further analyze $Pr(G)Pr(G_1|G)Pr(G_2|G, \mathbf{\Pi_0})$, we define $G^*_{\pi_0}$ as the graph which has the smallest number of edges in $\mathcal{G}_{\mathbf{\Pi}}$, i.e., $G^*_{\pi_0} = (V, E_1 \cup \pi_0(E_1))$, where $\pi_0(E_1) = \{(\pi_0(i), \pi_0(j))|(i,j) \in E_1\}$ and set $E^*_{\pi_0}$ as the edge set of $G^*_{\pi_0}$, and $E^{*ij}_{\pi_0}$ as the indicator variable between nodes i and j. By these definitions, we explicitly express that

$$\begin{aligned}&\sum_{G\in\mathcal{G}_{\mathbf{\Pi}}} Pr(G)Pr(G_1|G)Pr(G_2|G, \mathbf{\Pi_0}) \\ &\qquad \sim \sum_{i<j}^{n} E^{*ij}_{\pi_0} \log\left(\frac{p_{\boldsymbol{C_i}\boldsymbol{C_j}}(1-s_1)(1-s_2)}{1 - p_{\boldsymbol{C_i}\boldsymbol{C_j}}(s_1+s_2-s_1s_2)}\right)\end{aligned} \tag{7}$$

Ultimately, by analyzing the parameter $E^{*ij}_{\pi_0}$, we derive that

$$\begin{aligned}\hat{\mathbf{\Pi}} &= \arg\min_{\mathbf{\Pi}\in\mathbf{\Pi^n}} \sum_{\mathbf{\Pi_0}\in\mathbf{\Pi^n}} ||\mathbf{\Pi} - \mathbf{\Pi_0}||_F^2 \sum_{G\in\mathcal{G}_{\mathbf{\Pi}}} Pr(G, \mathbf{\Pi_0}|G_1, G_2, \boldsymbol{\theta}) \\ &\sim \arg\min_{\mathbf{\Pi}\in\mathbf{\Pi^n}} \sum_{\mathbf{\Pi_0}\in\mathbf{\Pi^n}} ||\mathbf{\Pi} - \mathbf{\Pi_0}||_F^2 \cdot \\ &\qquad \sum_{i<j}^{n} E^{*ij}_{\pi_0} \log\left(\frac{p_{\boldsymbol{C_i}\boldsymbol{C_j}}(1-s_1)(1-s_2)}{1 - p_{\boldsymbol{C_i}\boldsymbol{C_j}}(s_1+s_2-s_1s_2)}\right) \\ &= \arg\max_{\mathbf{\Pi}\in\mathbf{\Pi^n}} \sum_{\mathbf{\Pi_0}\in\mathbf{\Pi^n}} ||\mathbf{\Pi} - \mathbf{\Pi_0}||_F^2 ||\mathbf{W} \circ (\mathbf{\Pi_0}\mathbf{A} - \mathbf{B}\mathbf{\Pi_0})||_F^2.\end{aligned} \tag{8}$$

□

Remark: Additionally, to simplify the form of $||\mathbf{W} \circ (\mathbf{\Pi_0}\mathbf{A} - \mathbf{B}\mathbf{\Pi_0})||_F^2$, we use $\mathbf{\Pi_0}\hat{\mathbf{A}}$ to represent $\mathbf{W} \circ \mathbf{\Pi_0}\mathbf{A}$, and $\hat{\mathbf{B}}\mathbf{\Pi_0}$ to represent $\mathbf{W} \circ \mathbf{B}\mathbf{\Pi_0}$ [1]. Therefore we can rewrite

[1] We should clarify that we only provide a simpler form to represent $\mathbf{W} \circ \mathbf{\Pi_0}\mathbf{A}$ and $\mathbf{W} \circ \mathbf{B}\mathbf{\Pi_0}$, and it does NOT imply that $\mathbf{W} \circ \mathbf{A} = \hat{\mathbf{A}}$ and $\mathbf{W} \circ \mathbf{B} = \hat{\mathbf{B}}$ all the time. But some operations under this new notation still hold, for example, multiplying a permutation matrix does not change the value of the Frobenius norm, i.e., $||\mathbf{\Pi_0}\hat{\mathbf{A}} - \hat{\mathbf{B}}\mathbf{\Pi_0}||_F^2 = ||\mathbf{W} \circ (\mathbf{\Pi_0}\mathbf{A} - \mathbf{B}\mathbf{\Pi_0})||_F^2 = ||\mathbf{W} \circ \mathbf{\Pi_0^T}(\mathbf{\Pi_0}\mathbf{A} - \mathbf{B}\mathbf{\Pi_0})||_F^2 = ||\mathbf{W} \circ (\mathbf{A} - \mathbf{\Pi_0^T}\mathbf{B}\mathbf{\Pi_0})||_F^2$ and $||\mathbf{\Pi_0}\hat{\mathbf{A}} - \hat{\mathbf{B}}\mathbf{\Pi_0}||_F^2 = ||\mathbf{\Pi_0}\hat{\mathbf{A}}\mathbf{\Pi_0^T} - \hat{\mathbf{B}}||_F^2$. In Section V-A and Appendix D, we will discuss the condition under which $\mathbf{W} \circ \mathbf{A} = \hat{\mathbf{A}}$ and $\mathbf{W} \circ \mathbf{B} = \hat{\mathbf{B}}$.

the MMSE estimator in Eqn. (4) as

$$\hat{\mathbf{\Pi}} = \arg\max_{\mathbf{\Pi}\in\mathbf{\Pi^n}} g(\mathbf{\Pi}), \tag{9}$$

where $g(\mathbf{\Pi}) = \sum_{\mathbf{\Pi_0}\in\mathbf{\Pi^n}} ||\mathbf{\Pi}-\mathbf{\Pi_0}||_F^2 ||\mathbf{\Pi_0}\hat{\mathbf{A}}-\hat{\mathbf{B}}\mathbf{\Pi_0}||_F^2$.

C. Approximation of the MMSE estimator

As we have just stated above, the NP-hardness of MMSE problem urges us to find proper approximation for the original problem. Recall that MMSE involves all the possible true mappings, the number of which is $n!$, thus leading to fairly prohibitive computational cost. To tackle the difficulty, we firstly transform the original MMSE problem into a weighted-edge matching problem (WEMP), which, as we will define and present more details later, simplifies the form of objective function of the original MMSE problem and makes it tractable. Then we demonstrate that this transformation is valid, meaning that the solution of WEMP will not deviate much from the solution of the original MMSE problem by proving its high approximation ratio. Definition 4 provides the formal statement of WEMP.

Definition 4. ***(Weighted-Edge Matching Problem)*** *Given the adjacent matrices of G_1 and G_2, denoted as $\mathbf{A}$ and $\mathbf{B}$ respectively, we set $\mathbf{W}\circ\mathbf{\Pi A} = \mathbf{\Pi}\hat{\mathbf{A}}$ and $\mathbf{W}\circ\mathbf{B\Pi} = \hat{\mathbf{B}}\mathbf{\Pi}$ where $\mathbf{W}$ is the weight matrix, then the weight-edge matching problem is to find*

$$\tilde{\mathbf{\Pi}} = \arg\min_{\mathbf{\Pi}\in\mathbf{\Pi^n}} ||\mathbf{\Pi}\hat{\mathbf{A}}-\hat{\mathbf{B}}\mathbf{\Pi}||_F^2.$$

Hereinafter we discuss the following two aspects of WEMP:

- How do we transform from the original MMSE problem into WEMP?
- How is the validity of this transformation?

1) ***The Idea of Transformation****:* We intend to transform the original problem of solving the MMSE estimator into WEMP. The idea of this transformation can be interpreted in the following sense: for any fixed $\mathbf{\Pi}$, define a set $S_k(\mathbf{\Pi}), 0\le k\le n$, any element of which is an $n\times n$ permutation matrix $\mathbf{\Pi_0}$ such that $d(\mathbf{\Pi},\mathbf{\Pi_0}) = k$. When $k=0$, $d(\mathbf{\Pi},\mathbf{\Pi_0}) = 0$, thus $\mathbf{\Pi} = \mathbf{\Pi_0}$ and $S_0(\mathbf{\Pi}) = \{\mathbf{\Pi_0}\}$. What's more, if $\mathbf{\Pi}\neq\mathbf{\Pi_0}$, at least one node pair will be mismatched, thus $k\ge 2$ and $S_1(\mathbf{\Pi}) = \emptyset$. Then we can transform the original problem as

$$\hat{\mathbf{\Pi}} = \arg\max_{\mathbf{\Pi}\in\mathbf{\Pi^n}} \sum_{k=2}^{n} k\left(\sum_{\mathbf{\Pi_0}\in S_k(\mathbf{\Pi})} ||\mathbf{\Pi_0}\hat{\mathbf{A}}-\hat{\mathbf{B}}\mathbf{\Pi_0}||_F^2\right). \tag{10}$$

Based on Eqn. (10), we propose our idea of transforming it into WEMP. To present our idea clearly, we divide our analysis into three parts; First we analyze a single term, $||\mathbf{\Pi_0}\hat{\mathbf{A}}-\hat{\mathbf{B}}\mathbf{\Pi_0}||_F^2$, where $\mathbf{\Pi_0}\in S_2(\mathbf{\Pi})$; Then we analyze $\mathbf{\Pi_0}\in S_k(\mathbf{\Pi})$ based on the analysis of $\mathbf{\Pi_0}\in S_2(\mathbf{\Pi})$; Finally we analyze the R.H.S of Eqn. (10) based on Sequence Inequality. The detailed analysis of these three parts are unfolded in Appendix B and in the sequel we provide the sketch of them:

i. Analysis of $||\mathbf{\Pi_0}\hat{\mathbf{A}}-\hat{\mathbf{B}}\mathbf{\Pi_0}||_F^2$ where $\mathbf{\Pi_0}\in S_2(\mathbf{\Pi})$.

Note that any permutation in $S_2(\mathbf{\Pi})$ only causes matching error on one pair of nodes. We consider one specific $\mathbf{\Pi_0}\in S_2(\tilde{\mathbf{\Pi}})$, which differs from $\tilde{\mathbf{\Pi}}$ only in the i_{th} and j_{th} rows. Since G_1 and G_2 are independently sampled from G, $\mathbf{A}$ and $\mathbf{B}$ are conditionally independent. Thus, we can derive that

$$\mathbf{E_{A,B}}(||\mathbf{\Pi_0}\hat{\mathbf{A}}-\hat{\mathbf{B}}\mathbf{\Pi_0}||_F^2 - ||\tilde{\mathbf{\Pi}}\hat{\mathbf{A}}-\hat{\mathbf{B}}\tilde{\mathbf{\Pi}}||_F^2) = 2\sum_{k\neq i,j}^{n}\Delta_{i,j,k,\pi_0},$$

where

$$\Delta_{i,j,k,\pi_0} = (w_{ik}-2w_{ik}s_2-w_{jk})s_1 p_{\boldsymbol{C_i C_k}}$$
$$-(w_{ik}-w_{jk}+2w_{jk}s_2)s_1 p_{\boldsymbol{C_j C_k}} + 2(w_{ik}+w_{jk})s_1 s_2 p_{\boldsymbol{C_i C_k}} p_{\boldsymbol{C_j C_k}}.$$

This reflects that the difference between matrices $\mathbf{\Pi_0}\hat{\mathbf{A}}-\hat{\mathbf{B}}\mathbf{\Pi_0}$ and $\tilde{\mathbf{\Pi}}\hat{\mathbf{A}}-\hat{\mathbf{B}}\tilde{\mathbf{\Pi}}$ is composed of the differences between community pairs $\{i,k\}$ and $\{j,k\}$. We further denote $\hat{\Delta} = \mathbf{E}_{i,j,\pi_0}(\Delta_{i,j,\pi_0})$ and note that $\mathbf{E_{A,B}}(||\mathbf{\Pi_0}\hat{\mathbf{A}}-\hat{\mathbf{B}}\mathbf{\Pi_0}||_F^2 - ||\tilde{\mathbf{\Pi}}\hat{\mathbf{A}}-\hat{\mathbf{B}}\tilde{\mathbf{\Pi}}||_F^2) > 0$ since $\tilde{\mathbf{\Pi}}$ is the minimizer of $||\mathbf{\Pi_0}\hat{\mathbf{A}}-\hat{\mathbf{B}}\mathbf{\Pi_0}||_F^2$. Therefore $\hat{\Delta} = \mathbf{E}_{i,j,\pi_0}(\Delta_{i,j,\pi_0}) > 0$.

ii. Analysis of $\sum_{\mathbf{\Pi_0}\in S_k(\mathbf{\Pi})} ||\mathbf{\Pi_0}\hat{\mathbf{A}}-\hat{\mathbf{B}}\mathbf{\Pi_0}||_F^2$.

We first focus on $S_k(\mathbf{\Pi_0})$, and count the number of elements in $S_k(\mathbf{\Pi_0})$, denoted as $|S_k|$. We can obtain the relationship between $|S_k|$ and $|S_{k-1}|$ when $k\ge 2$ as

$$|S_k| = C_n^k|T_k| \ge (k-1)\frac{C_n^k}{C_n^{k-1}}|S_{k-1}| = (1-\frac{1}{k})(n-k+1)|S_{k-1}|,$$

where the detailed derivation is deferred to Appendix B.

Then we consider $\mathbf{\Pi_0}\in S_k(\mathbf{\Pi})$. Note that for any $\mathbf{\Pi_0}\in S_k(\mathbf{\Pi})$, there are k rows and columns that may cause the difference between $||\mathbf{W}\circ(\mathbf{\Pi_0 A \Pi_0^T}-\mathbf{B})||_F^2$ and $||\mathbf{W}\circ(\tilde{\mathbf{\Pi}}\mathbf{A}\tilde{\mathbf{\Pi}}^{\mathbf{T}}-\mathbf{B})||_F^2$. We can discover that for any $\mathbf{\Pi_0}\in S_k(\mathbf{\Pi})$, the number of node pairs (i,j) which may influence the difference between $||\mathbf{W}\circ(\mathbf{\Pi_0 A \Pi_0^T}-\mathbf{B})||_F^2$ and $||\mathbf{W}\circ(\tilde{\mathbf{\Pi}}\mathbf{A}\tilde{\mathbf{\Pi}}^{\mathbf{T}}-\mathbf{B})||_F^2$ is approximately $\sum_{i=1}(n-i) = \frac{(2n-k-1)k}{2}$. Thus, denoting N_k as the number of such node pairs for all $\mathbf{\Pi_0}\in S_k(\mathbf{\Pi})$, we can obtain

$$N_k = \frac{(2n-k-1)k}{2}|S_k| \ge (n-k+1)\frac{2n-k-1}{2n-k}N_{k-1}.$$

Therefore on average, we have

$$\begin{aligned}\sum_{\mathbf{\Pi_0}\in S_k} ||\mathbf{\Pi_0}\hat{\mathbf{A}}-\hat{\mathbf{B}}\mathbf{\Pi_0}||_F^2 &= N_k\hat{\Delta}\\ &\ge (n-k+1)\sum_{\mathbf{\Pi_0}\in S_{k-1}} ||\mathbf{\Pi_0}\hat{\mathbf{A}}-\hat{\mathbf{B}}\mathbf{\Pi_0}||_F^2,\end{aligned} \tag{11}$$

where $k\le n$ and $\hat{\Delta} = \mathbf{E}_{i,j,\pi_0}(\Delta_{i,j,\pi_0})$.

Thus, we can claim that in average sense, if $k_1 > k_2$, then

$$\sum_{\mathbf{\Pi_0}\in S_{k_1}} ||\mathbf{\Pi_0}\hat{\mathbf{A}}-\hat{\mathbf{B}}\mathbf{\Pi_0}||_F^2 > \sum_{\mathbf{\Pi_0}\in S_{k_2}} ||\mathbf{\Pi_0}\hat{\mathbf{A}}-\hat{\mathbf{B}}\mathbf{\Pi_0}||_F^2. \tag{12}$$

iii. Maximum Value Under Sequence Inequality.

Based on the analysis above, we set $\mathbf{\Pi} = \tilde{\mathbf{\Pi}}$, if $\mathbf{\Pi_0}\in S_0(\tilde{\mathbf{\Pi}})$, $||\tilde{\mathbf{\Pi}}-\mathbf{\Pi_0}||_F^2$ is the minimum value. Moreover, $\sum_{\mathbf{\Pi_0}\in S_0(\tilde{\mathbf{\Pi}})} ||\mathbf{\Pi_0}\hat{\mathbf{A}}-\hat{\mathbf{B}}\mathbf{\Pi_0}||_F^2 = ||\tilde{\mathbf{\Pi}}\hat{\mathbf{A}}-\hat{\mathbf{B}}\tilde{\mathbf{\Pi}}||_F^2$ is also the minimum value in the set

$$\left\{\begin{matrix}\sum_{\mathbf{\Pi_0}\in S_0(\tilde{\mathbf{\Pi}})} ||(\mathbf{\Pi_0}\hat{\mathbf{A}}-\hat{\mathbf{B}}\mathbf{\Pi_0})||_F^2, \sum_{\mathbf{\Pi_0}\in S_2(\tilde{\mathbf{\Pi}})} ||(\mathbf{\Pi_0}\hat{\mathbf{A}}-\hat{\mathbf{B}}\mathbf{\Pi_0})||_F^2,\\ \sum_{\mathbf{\Pi_0}\in S_3(\tilde{\mathbf{\Pi}})} ||(\mathbf{\Pi_0}\hat{\mathbf{A}}-\hat{\mathbf{B}}\mathbf{\Pi_0})||_F^2, ..., \sum_{\mathbf{\Pi_0}\in S_n(\tilde{\mathbf{\Pi}})} ||(\mathbf{\Pi_0}\hat{\mathbf{A}}-\hat{\mathbf{B}}\mathbf{\Pi_0})||_F^2\end{matrix}\right\}.$$

Thus according to sequence inequality, we know that in average case, by setting $\mathbf{\Pi}$ in the original MMSE objective

function of Eqn. (9) to equal to $\tilde{\mathbf{\Pi}}$, the minimizer of WEMP, this original MMSE objective function reaches its largest value [2] under Sequence Inequality.

However, as we noted earlier, we can only transform the original MMSE problem into WEMP in an average case of network structures. This implies that the transformation is not necessarily the best approximation of a single network structure. In the following we further analyze the validity of this transformation in a possible network structure by showing the approximation ratio of our transformation is large (at least larger than 0.5).

*2) **The Validity of Transformation**:* As we have stated above, $\tilde{\mathbf{\Pi}}$ does not necessarily achieve the maximum of the original MMSE problem for a specific network structure. It indicates that there may exist error in $g(\tilde{\mathbf{\Pi}})$ and $g(\hat{\mathbf{\Pi}})$, where $g(\hat{\mathbf{\Pi}})$ is the maximum value of the original MMSE objective function and $g(\tilde{\mathbf{\Pi}})$ is the value of MMSE objective function when $\mathbf{\Pi}$ equals to the minimizer of WEMP. Theorem 2 shows that under the mild condition indicated by Inequality (11), we can get approximation ratio $g(\tilde{\mathbf{\Pi}})/g(\hat{\mathbf{\Pi}})$ larger than 0.5, which, to some extent, makes our estimation reasonable.

Theorem 2. *Given the published graph G_1, the auxiliary graph G_2, the parameter set $\boldsymbol{\theta}$ and the weight matrix $\mathbf{W}$, in average case we have the approximation ratio $g(\tilde{\mathbf{\Pi}})/g(\hat{\mathbf{\Pi}})$ larger than* 0.5.

Proof. We present the sketch of the proof here and defer the complete version to the Appendix C.

First we derive the upperbound of $\frac{g(\hat{\mathbf{\Pi}})-g(\tilde{\mathbf{\Pi}})}{g(\tilde{\mathbf{\Pi}})}$:

$$\frac{g(\hat{\mathbf{\Pi}})-g(\tilde{\mathbf{\Pi}})}{g(\tilde{\mathbf{\Pi}})} \leq \frac{2\beta n \sum_{\mathbf{\Pi_0}\in\mathbf{\Pi^n}} ||\mathbf{\Pi_0}\tilde{\mathbf{A}}-\tilde{\mathbf{B}}\mathbf{\Pi_0}||_F^2}{\sum_{\mathbf{\Pi_0}\in\mathbf{\Pi^n}} ||\tilde{\mathbf{\Pi}}-\mathbf{\Pi_0}||_F^2 ||\mathbf{\Pi_0}\tilde{\mathbf{A}}-\tilde{\mathbf{B}}\mathbf{\Pi_0}||_F^2}. \tag{13}$$

where $||\tilde{\mathbf{\Pi}}-\hat{\mathbf{\Pi}}||_F^2 = 2\beta n$ and $\beta \in [0,1]$ is the ratio between the number of mistakenly matched nodes and that of all the nodes.

Then we divide $\sum_{\mathbf{\Pi_0}\in\mathbf{\Pi^n}} ||\mathbf{\Pi_0}\hat{\mathbf{A}}-\hat{\mathbf{B}}\mathbf{\Pi_0}||_F^2$ into two parts:

$$D_1 = \sum_{k\leq\rho n}\sum_{\mathbf{\Pi_0}\in\mathbf{\Pi^n}} ||\mathbf{\Pi_0}\hat{\mathbf{A}}-\hat{\mathbf{B}}\mathbf{\Pi_0}||_F^2;$$

$$D_2 = \sum_{\rho n<k\leq n}\sum_{\mathbf{\Pi_0}\in\mathbf{\Pi^n}} ||\mathbf{\Pi_0}\hat{\mathbf{A}}-\hat{\mathbf{B}}\mathbf{\Pi_0}||_F^2.$$

where ρ is any real number in $[0,1]$ and we assume that ρn is an integer, since if it is not an integer, we can easily modify it by rounding operation.

For D_1 and D_2, in average case we can obtain $D_1 \leq (2n)^{\rho n}$ and $D_2 \geq (1-\rho)n\frac{n!}{((1-\rho)n)!} = \frac{n!}{((1-\rho)n-1)!}$.

Note that if we set $\rho = c_0 \to 1_-$, the parameter $(1-\rho)n$ can be upper bounded by a constant c_1 and

$$D_2 \geq \frac{n!}{(c_1-1)!} = cn! \sim c\sqrt{2\pi n}(\frac{n}{e})^n,$$

[2]We set $k_1 > k_2$, then $\sum_{\mathbf{\Pi_0}\in S_{k_1}} ||\mathbf{\Pi_0}\hat{\mathbf{A}}-\hat{\mathbf{B}}\mathbf{\Pi_0}||_F^2 > \sum_{\mathbf{\Pi_0}\in S_{k_2}} ||\mathbf{\Pi_0}\hat{\mathbf{A}}-\hat{\mathbf{B}}\mathbf{\Pi_0}||_F^2$, and $||\tilde{\mathbf{\Pi}}-\mathbf{\Pi_0}||_F^2$ where $\mathbf{\Pi_0} \in S_{k_1}(\tilde{\mathbf{\Pi}})$ is larger than $||\tilde{\mathbf{\Pi}}-\mathbf{\Pi_0}||_F^2$ where $\mathbf{\Pi_0} \in S_{k_2}(\tilde{\mathbf{\Pi}})$. Thus, $\sum_{\mathbf{\Pi_0}\in\Pi^n} ||\mathbf{\Pi}-\mathbf{\Pi_0}||_F^2 ||\mathbf{W}\circ(\mathbf{\Pi_0}\mathbf{A}-\mathbf{B}\mathbf{\Pi_0})||_F^2 = \sum_{k=0}^n 2k\sum_{\mathbf{\Pi_0}\in S_k(\tilde{\mathbf{\Pi}})} ||\mathbf{\Pi_0}\hat{\mathbf{A}}-\hat{\mathbf{B}}\mathbf{\Pi_0}||_F^2$ reaches its largest value.

where c is a constant and the last step holds due to the Stirling's formula. Therefore we can upper bound $\frac{D_2}{D_1}$ as

$$\frac{D_2}{D_1} \geq c\frac{\sqrt{2\pi n}(\frac{n}{e})^n}{(2n)^{\rho n}} = c\sqrt{2\pi n}\left(\frac{n^{1-\rho}}{2^\rho e}\right)^n.$$

If ρ is a constant and $\rho \to 1$, we can find that when $n \to \infty$, D_2 is of higher order of n than D_1. Therefore, we verify that $\sum_{\rho n<k\leq n}\sum_{\mathbf{\Pi_0}\in\mathbf{\Pi^n}} ||\tilde{\mathbf{\Pi}}-\mathbf{\Pi_0}||_F^2 ||\mathbf{\Pi_0}\hat{\mathbf{A}}-\hat{\mathbf{B}}\mathbf{\Pi_0}||_F^2$ is of higher order of n than $\sum_{k\leq\rho n}\sum_{\mathbf{\Pi_0}\in\mathbf{\Pi^n}} ||\tilde{\mathbf{\Pi}}-\mathbf{\Pi_0}||_F^2 ||\mathbf{\Pi_0}\hat{\mathbf{A}}-\hat{\mathbf{B}}\mathbf{\Pi_0}||_F^2$, since for $k_1 > \rho n$ and $k_2 < \rho n$, $\mathbf{\Pi'_1} \in S_{k_1}(\tilde{\mathbf{\Pi}})$ and $\mathbf{\Pi'_2} \in S_{k_2}(\tilde{\mathbf{\Pi}})$, we have $||\mathbf{\Pi'_1}-\tilde{\mathbf{\Pi}}||_F^2 \geq ||\mathbf{\Pi'_2}-\tilde{\mathbf{\Pi}}||_F^2$. Thus we can obtain

$$\begin{aligned}
&\frac{2\beta n\sum_{\mathbf{\Pi_0}\in\mathbf{\Pi^n}} ||\mathbf{\Pi_0}\hat{\mathbf{A}}-\hat{\mathbf{B}}\mathbf{\Pi_0}||_F^2}{\sum_{\mathbf{\Pi_0}\in\mathbf{\Pi^n}} ||\tilde{\mathbf{\Pi}}-\mathbf{\Pi_0}||_F^2 ||\mathbf{\Pi_0}\hat{\mathbf{A}}-\hat{\mathbf{B}}\mathbf{\Pi_0}||_F^2} \\
&\approx \frac{2\beta n\sum_{\rho n<k\leq n}\sum_{\mathbf{\Pi_0}\in S_k(\tilde{\mathbf{\Pi}})} ||\mathbf{\Pi_0}\hat{\mathbf{A}}-\hat{\mathbf{B}}\mathbf{\Pi_0}||_F^2}{\sum_{\rho n<k\leq n}\sum_{\mathbf{\Pi_0}\in S_k(\tilde{\mathbf{\Pi}})} ||\mathbf{\Pi_0}-\tilde{\mathbf{\Pi}}||_F^2 ||\mathbf{\Pi_0}\hat{\mathbf{A}}-\hat{\mathbf{B}}\mathbf{\Pi_0}||_F^2} \\
&\leq \frac{2\beta n}{2\rho n}\frac{\sum_{\rho n<k\leq n}\sum_{\mathbf{\Pi_0}\in S_k(\tilde{\mathbf{\Pi}})} ||\mathbf{\Pi_0}\hat{\mathbf{A}}-\hat{\mathbf{B}}\mathbf{\Pi_0}||_F^2}{\sum_{\rho n<k\leq n}\sum_{\mathbf{\Pi_0}\in S_k(\tilde{\mathbf{\Pi}})} ||\mathbf{\Pi_0}\hat{\mathbf{A}}-\hat{\mathbf{B}}\mathbf{\Pi_0}||_F^2} = \frac{\beta}{\rho}.
\end{aligned}$$

We finally obtain the approximation ratio

$$\frac{g(\tilde{\mathbf{\Pi}})}{g(\hat{\mathbf{\Pi}})} \geq \frac{1}{1+\frac{\beta}{\rho}} \approx \frac{1}{1+\beta} \geq \frac{1}{2}.$$

□

V. Algorithmic Aspect of De-anonymization Problem

In this section, we show that WEMP is of significant advantages in seedless de-anonymization since it resolves the tension between *optimality* and *complexity*. For optimality, We prove the good performance of solving WEMP that the result makes the node mapping error (NME) negligible in large social networks under mild conditions, facilitated by higher overlapping strength; For complexity, the optimal mapping of WEMP, $\tilde{\mathbf{\Pi}}$, can be perfectly sought algorithmically by our convex-concave based de-anonymization algorithm (CBDA).

A. Optimality: WEMP Returns Negligible NME

Recall that our aim is to minimize NME in expectation, a natural question arises: *what is the NME between the optimal solution $\tilde{\mathbf{\Pi}}$ of WEMP and the true permutation matrix $\mathbf{\Pi_0}$?* The answer will indicate the ability of solving WEMP in enhancing the mapping accuracy. To answer this question, we define the *relative NME* as follows.

Definition 5. ***(Relative NME)*** *The relative NME is defined as the ratio of mismatched nodes, i.e.,* $\frac{||\tilde{\mathbf{\Pi}}-\mathbf{\Pi_0}||_F^2}{||\mathbf{\Pi_0}||_F^2}$.

We then demonstrate in the following that under mild conditions, the relative NME vanishes to 0 as $n \to \infty$. The relative NME is actually the proportion of mismatched nodes in the network. Our result implies that in a large-sized network, the NME caused by $\tilde{\mathbf{\Pi}}$ is negligible compared with $|V| = n$. Furthermore, we surprisingly find that the conditions are facilitated under higher overlapping strength,

explicitly delineating benefits brought by overlapping communities in NME reduction. Theorem 3 formally presents our result mentioned above. Before that, we provide Lemma 1, whose proof is provided in Appendix D, as a prerequisite in proving Theorem 3.

Lemma 1. *Suppose that the permutation matrix $\mathbf{\Pi}$ ensures that all nodes are mapped with those who have same community structures, i.e., for every i, j, if $\mathbf{\Pi}(i,j) = 1$, $\boldsymbol{C_i} = \boldsymbol{C_j}$. On this condition, there will be $\hat{\mathbf{A}} = \mathbf{W} \circ \mathbf{A}$, $\hat{\mathbf{B}} = \mathbf{W} \circ \mathbf{B}$ and $||\mathbf{W} \circ (\mathbf{\Pi A \Pi^T} - \mathbf{B})||_F = ||\mathbf{\Pi \hat{A} \Pi^T} - \hat{\mathbf{B}}||_F$.*

Remark: Note that there are no differences between $||\mathbf{\Pi \hat{A} \Pi^T} - \hat{\mathbf{B}}||_F$ and $||\hat{\mathbf{A}} - \mathbf{\Pi \hat{B} \Pi^T}||_F$ since we can simply find that $\mathbf{\Pi \Pi^T} = \mathbf{I}$. Therefore, we do not distinguish the forms $||\mathbf{\Pi \hat{A} \Pi^T} - \hat{\mathbf{B}}||_F$ and $||\hat{\mathbf{A}} - \mathbf{\Pi \hat{B} \Pi^T}||_F$ anymore.

We can then introduce one of the most important results in this paper as follows.

Theorem 3. *Given $G_1(V_1, E_1, \mathbf{A})$, $G_2(V_2, E_2, \mathbf{B})$, $\boldsymbol{\theta}$ and $\mathbf{W}$. Let $K = \min_{s,t,j}\{(p_{\boldsymbol{C_s C_j}} + p_{\boldsymbol{C_t C_j}}) \min\{s_1, s_2\}\}$, $L = \max_{s,t,j}\{[(p_{\boldsymbol{C_s C_j}} + p_{\boldsymbol{C_t C_j}}) \max\{s_1, s_2\}]^2\}$. If (i) $\frac{L}{K} = o(1)$; (ii) $\frac{||\hat{\mathbf{A}} - \mathbf{\Pi_0 \hat{B} \Pi_0^T}||_F^2}{||\hat{\mathbf{A}} - \tilde{\mathbf{\Pi}} \hat{\mathbf{B}} \tilde{\mathbf{\Pi}}^{\mathbf{T}}||_F^2} = \Omega(1)$; (iii) $||\hat{\mathbf{A}} - \mathbf{\Pi_0 \hat{B} \Pi_0^T}||_F^2 = o(Kn^2)$; (iv) $\mathbf{\Pi_0}$ and $\tilde{\mathbf{\Pi}}$ ensures that all nodes are mapped with those who have same community structures, then as $n \to \infty$, $\frac{||\tilde{\mathbf{\Pi}} - \mathbf{\Pi_0}||_F^2}{||\mathbf{\Pi_0}||_F^2} \to 0$.*

Proof. The proof is composed of the following four steps:

i. Upper bounding $||\tilde{\mathbf{\Pi}} - \mathbf{\Pi_0}||_F$ by $||(\tilde{\mathbf{\Pi}} - \mathbf{\Pi_0})\hat{\mathbf{B}}||_F$;

ii. Finding the relationship between $||(\tilde{\mathbf{\Pi}} - \mathbf{\Pi_0})\hat{\mathbf{B}}||_F$ and $\mathbf{tr}((\tilde{\mathbf{\Pi}} - \mathbf{\Pi_0})\hat{\mathbf{B}}(\tilde{\mathbf{\Pi}} - \mathbf{\Pi_0})^{\mathbf{T}}\hat{\mathbf{A}})$;

iii. Upper bounding $\mathbf{tr}((\tilde{\mathbf{\Pi}} - \mathbf{\Pi_0})\hat{\mathbf{B}}(\tilde{\mathbf{\Pi}} - \mathbf{\Pi_0})^{\mathbf{T}}\hat{\mathbf{A}})$;

iv. Upper bounding $\frac{||\mathbf{\Pi_0} - \tilde{\mathbf{\Pi}}||_F^2}{||\mathbf{\Pi_0}||_F^2}$.

i. Upper bounding $||\tilde{\mathbf{\Pi}} - \mathbf{\Pi_0}||_F$ by $||(\tilde{\mathbf{\Pi}} - \mathbf{\Pi_0})\hat{\mathbf{B}}||_F$.

For the i_{th} row of $(\mathbf{\Pi_0} - \tilde{\mathbf{\Pi}})$, we set $\pi_0(i) = s$ and $\tilde{\pi}(i) = t$. If $s = t$, then the i_{th} row of $(\mathbf{\Pi_0} - \tilde{\mathbf{\Pi}})\hat{\mathbf{B}}$ is a zero vector; else the i_{th} row of $(\mathbf{\Pi_0} - \tilde{\mathbf{\Pi}})\hat{\mathbf{B}}$ is $(\mathbf{B}_{s1} - \mathbf{B}_{t1}, \mathbf{B}_{s2} - \mathbf{B}_{t2}, \cdots, \mathbf{B}_{sn} - \mathbf{B}_{tn})$. For an element, $([(\mathbf{\Pi_0} - \tilde{\mathbf{\Pi}})\hat{\mathbf{B}}]_{ij})^2 = (\mathbf{B}_{sj} - \mathbf{B}_{tj})^2$. Taking the expectation on both sides, we can derive that

$$\mathbf{E_B}[(\mathbf{\Pi_0} - \tilde{\mathbf{\Pi}})\hat{\mathbf{B}}]_{ij}^2 = (p_{\boldsymbol{C_s C_j}} + p_{\boldsymbol{C_t C_j}} - 2p_{\boldsymbol{C_s C_j}} p_{\boldsymbol{C_t C_j}} s_2)s_2,$$

where $\mathbf{E_B}$ means taking expectation on every element in $\mathbf{B}$. By summing up all the rows and columns,

$$\begin{aligned}
&||(\mathbf{\Pi_0} - \tilde{\mathbf{\Pi}})\hat{\mathbf{B}}||_F^2 = \mathbf{E}\sum_{i=1}^{n}\sum_{j=1}^{n}[(\mathbf{\Pi_0} - \tilde{\mathbf{\Pi}})\hat{\mathbf{B}}]_{ij}^2 \\
&= \sum_{i=1}^{n} \mathbb{1}\{\pi_0(i) \neq \tilde{\pi}(i)\} \sum_{j=1}^{n}(p_{\boldsymbol{C_s C_j}} + p_{\boldsymbol{C_t C_j}} - 2p_{\boldsymbol{C_s C_j}} p_{\boldsymbol{C_t C_j}} s_2)s_2 \\
&\geq \sum_{i=1}^{n} n\mathbb{1}\{\pi_0(i) \neq \tilde{\pi}(i)\} \min_j(p_{\boldsymbol{C_s C_j}} + p_{\boldsymbol{C_t C_j}} - 2p_{\boldsymbol{C_s C_j}} p_{\boldsymbol{C_t C_j}} s_2)s_2,
\end{aligned}$$

Setting $K_2 = \min_{s,t,j}(p_{\boldsymbol{C_s C_j}} + p_{\boldsymbol{C_t C_j}} - 2p_{\boldsymbol{C_s C_j}} p_{\boldsymbol{C_t C_j}} s_2)s_2$. Note that $||(\mathbf{\Pi_0} - \tilde{\mathbf{\Pi}})||_F^2 = 2\sum_{i=1}^{n} \mathbf{1}\{\pi_0(i) \neq \tilde{\pi}(i)\}$, we have

$$||\mathbf{\Pi_0} - \tilde{\mathbf{\Pi}}||_F^2 \leq \frac{2}{nK_2}||(\mathbf{\Pi_0} - \tilde{\mathbf{\Pi}})\hat{\mathbf{B}}||_F^2. \tag{14}$$

Similarly we can replace $\hat{\mathbf{B}}$ with $\hat{\mathbf{A}}$ and replace s_2 with s_1 to obtain K_1, which provides $K = \min\{K_1, K_2\}$.

ii. $||(\mathbf{\Pi_0} - \tilde{\mathbf{\Pi}})\hat{\mathbf{B}}||_F$ and $\mathbf{tr}((\tilde{\mathbf{\Pi}} - \mathbf{\Pi_0})\hat{\mathbf{B}}((\tilde{\mathbf{\Pi}} - \mathbf{\Pi_0})^{\mathbf{T}})\hat{\mathbf{A}})$.

Note that

$$\begin{aligned}
||(\mathbf{\Pi_0} - \tilde{\mathbf{\Pi}})\hat{\mathbf{B}}||_F &= ||\hat{\mathbf{B}}(\mathbf{\Pi_0} - \tilde{\mathbf{\Pi}})^{\mathbf{T}}||_F = ||\tilde{\mathbf{\Pi}}\hat{\mathbf{B}}(\mathbf{\Pi_0} - \tilde{\mathbf{\Pi}})^{\mathbf{T}}||_F \\
&\leq ||(\tilde{\mathbf{\Pi}}\hat{\mathbf{B}}\mathbf{\Pi_0} - \hat{\mathbf{A}}) - (\tilde{\mathbf{\Pi}}\hat{\mathbf{B}}\tilde{\mathbf{\Pi}} - \hat{\mathbf{A}})||_F \\
&\leq ||\tilde{\mathbf{\Pi}}\hat{\mathbf{B}}\tilde{\mathbf{\Pi}}^{\mathbf{T}} - \hat{\mathbf{A}}||_F + ||\tilde{\mathbf{\Pi}}\hat{\mathbf{B}}\mathbf{\Pi_0^T} - \hat{\mathbf{A}}||_F,
\end{aligned}$$

where the second equation holds since the permutation matrix $\tilde{\mathbf{\Pi}}$ keeps invariant of Frobenius norm, and the second inequality holds due to the triangular inequality of Frobenius norm. Then we obtain

$$||(\mathbf{\Pi_0} - \tilde{\mathbf{\Pi}})\hat{\mathbf{B}}||_F^2 \leq 2(||\tilde{\mathbf{\Pi}}\hat{\mathbf{B}}\tilde{\mathbf{\Pi}}^{\mathbf{T}} - \hat{\mathbf{A}}||_F^2 + ||\tilde{\mathbf{\Pi}}\hat{\mathbf{B}}\mathbf{\Pi_0^T} - \hat{\mathbf{A}}||_F^2).$$

For the term $||\tilde{\mathbf{\Pi}}\hat{\mathbf{B}}\mathbf{\Pi_0^T} - \hat{\mathbf{A}}||_F^2$,

$$\begin{aligned}
&||\tilde{\mathbf{\Pi}}\hat{\mathbf{B}}\mathbf{\Pi_0^T} - \hat{\mathbf{A}}||_F^2 = \mathbf{tr}((\tilde{\mathbf{\Pi}}\hat{\mathbf{B}}\mathbf{\Pi_0^T} - \hat{\mathbf{A}})^{\mathbf{T}}(\tilde{\mathbf{\Pi}}\hat{\mathbf{B}}\mathbf{\Pi_0^T} - \hat{\mathbf{A}})) \\
=&\mathbf{tr}(\hat{\mathbf{A}}^{\mathbf{T}}\hat{\mathbf{A}}) + \mathbf{tr}(\hat{\mathbf{B}}^{\mathbf{T}}\hat{\mathbf{B}}) - 2\mathbf{tr}(\mathbf{\Pi_0}\hat{\mathbf{B}}\tilde{\mathbf{\Pi}}^{\mathbf{T}}\hat{\mathbf{A}}) \\
=&||\hat{\mathbf{A}}||_F^2 + ||\hat{\mathbf{B}}||_F^2 - 2\mathbf{tr}(\mathbf{\Pi_0}\hat{\mathbf{B}}\tilde{\mathbf{\Pi}}^{\mathbf{T}}\hat{\mathbf{A}}) \\
=&\frac{1}{2}(||\tilde{\mathbf{\Pi}}\hat{\mathbf{B}}\tilde{\mathbf{\Pi}}^{\mathbf{T}} - \hat{\mathbf{A}}||_F^2 + ||\mathbf{\Pi_0}\hat{\mathbf{B}}\mathbf{\Pi_0^T} - \hat{\mathbf{A}}||_F^2) \\
&+ \mathbf{tr}(\mathbf{\Pi_0}\hat{\mathbf{B}}\mathbf{\Pi_0^T}\hat{\mathbf{A}}) + \mathbf{tr}(\tilde{\mathbf{\Pi}}\hat{\mathbf{B}}\tilde{\mathbf{\Pi}}^{\mathbf{T}}\hat{\mathbf{A}}) - 2\mathbf{tr}(\mathbf{\Pi_0}\hat{\mathbf{B}}\tilde{\mathbf{\Pi}}^{\mathbf{T}}\hat{\mathbf{A}}) \\
\leq&||\mathbf{\Pi_0}\hat{\mathbf{B}}\mathbf{\Pi_0}^T - \hat{\mathbf{A}}||_F^2 + \mathbf{tr}((\tilde{\mathbf{\Pi}} - \mathbf{\Pi_0})\hat{\mathbf{B}}((\tilde{\mathbf{\Pi}} - \mathbf{\Pi_0})^{\mathbf{T}})\hat{\mathbf{A}}),
\end{aligned} \tag{15}$$

the last inequality holds since $||\tilde{\mathbf{\Pi}}\hat{\mathbf{B}}\tilde{\mathbf{\Pi}}^{\mathbf{T}} - \hat{\mathbf{A}}||_F^2 \leq ||\mathbf{\Pi_0}\hat{\mathbf{B}}\mathbf{\Pi_0^T} - \hat{\mathbf{A}}||_F^2$.

iii. Upper Bound of $\mathbf{tr}((\tilde{\mathbf{\Pi}} - \mathbf{\Pi_0})\hat{\mathbf{B}}(\tilde{\mathbf{\Pi}} - \mathbf{\Pi_0})^{\mathbf{T}}\hat{\mathbf{A}})$.

Set $\mathbf{Z} = (\tilde{\mathbf{\Pi}} - \mathbf{\Pi_0})\hat{\mathbf{B}}(\tilde{\mathbf{\Pi}} - \mathbf{\Pi_0})^{\mathbf{T}}\hat{\mathbf{A}}$. For simplicity, we define $\mathbf{Y} = (\tilde{\mathbf{\Pi}} - \mathbf{\Pi_0})\hat{\mathbf{B}}$ and $\mathbf{X} = (\tilde{\mathbf{\Pi}} - \mathbf{\Pi_0})^{\mathbf{T}}\hat{\mathbf{A}}$, thus $\mathbf{Z} = \mathbf{YX}$. We focus on $\mathbf{tr}(\mathbf{Z})$. For any node i, when $\tilde{\mathbf{\Pi}}$ and $\mathbf{\Pi_0}$ map it to the same node, the ith column of $\tilde{\mathbf{\Pi}} - \mathbf{\Pi_0}$ is all-zero, hence $\mathbf{Z}_{ii} = 0$. Otherwise, for node i we assume that $\tilde{\mathbf{\Pi}}$ maps it to s and $\mathbf{\Pi_0}$ maps it to t, where $s \neq t$. We can obtain the i_{th} row of $\mathbf{Y}$ as $\mathbf{Y_{i\cdot}} = (\hat{\mathbf{B}}_{s1} - \hat{\mathbf{B}}_{t1}, \hat{\mathbf{B}}_{s2} - \hat{\mathbf{B}}_{t2}, \cdots, \hat{\mathbf{B}}_{sn} - \hat{\mathbf{B}}_{tn})$. Similarly, we can obtain the i_{th} column of $\mathbf{X}$ as $\mathbf{X_{\cdot i}} = (\hat{\mathbf{A}}_{p_1 1} - \hat{\mathbf{A}}_{q_1 1}, \hat{\mathbf{A}}_{p_2 2} - \hat{\mathbf{A}}_{q_2 2}, \cdots, \hat{\mathbf{A}}_{p_n n} - \hat{\mathbf{A}}_{q_n n})^{\mathbf{T}}$, where $p_i(q_i)$ means the row index of the $1(-1)$ in the i_{th} column of $\tilde{\mathbf{\Pi}} - \mathbf{\Pi_0}$. If $\pi_0(j) = \tilde{\pi}(j)$, we simply set $\mathbf{X}_{ji} = 0$. Therefore $\mathbf{Z}_{ii}$, an element on diagonal of $\mathbf{Z}$, satisfies

$$\begin{aligned}
|\mathbf{Z}_{ii}| &= |\langle \mathbf{Y_{i\cdot}}, \mathbf{X_{\cdot i}} \rangle| \leq ||\mathbf{Y_{i\cdot}}||_F ||\mathbf{X_{\cdot i}}||_F \\
&\leq n \max_k |\hat{\mathbf{B}}_{sk} - \hat{\mathbf{B}}_{tk}| \max_\ell |\hat{\mathbf{A}}_{p_\ell \ell} - \hat{\mathbf{A}}_{q_\ell \ell}|.
\end{aligned} \tag{16}$$

Note that if we normalize w_{ij} to $[0, 1]$ by dividing $||\mathbf{W}||_F$, with no impact on $\tilde{\mathbf{\Pi}}$ since it is irrelevant with $||\mathbf{W}||_F$, then $|\mathbf{Z}_{ii}| \leq n$. Taking the expectation of $\mathbf{A}$ and $\mathbf{B}$ on both sides of Inequality (16), we can obtain that

$$\begin{aligned}
\mathbf{E_{A,B}}|\mathbf{Z}_{ii}| &= \mathbf{E_{A,B}}(\max_{s,t,k} |\hat{\mathbf{B}}_{sk} - \hat{\mathbf{B}}_{tk}| \max_{p,q,\ell} |\hat{\mathbf{A}}_{p_\ell \ell} - \hat{\mathbf{A}}_{q_\ell \ell}|) \\
&\leq \mathbf{E_{A,B}}(\max_{s,t,k} |\mathbf{B}_{sk} - \mathbf{B}_{tk}| \max_{p,q,\ell} |\mathbf{A}_{p_\ell \ell} - \mathbf{A}_{q_\ell \ell}|) \\
&\leq \max_{p,q,\ell}\{[(p_{\boldsymbol{C_s C_j}} + p_{\boldsymbol{C_t C_j}}) \max\{s_1, s_2\}]^2\} = L,
\end{aligned}$$

where the first inequality holds since for any s, t, k and the normalized weights $w_{sk}, w_{tk} \leq 1$, $|\hat{\mathbf{B}}_{sk} - \hat{\mathbf{B}}_{tk}| = |\sqrt{w_{sk}}\mathbf{B}_{sk} - \sqrt{w_{tk}}\mathbf{B}_{tk}| \leq |\mathbf{B}_{sk} - \mathbf{B}_{tk}|$, and $|\hat{\mathbf{A}}_{p_\ell \ell} - \hat{\mathbf{A}}_{q_\ell \ell}|$ is similar. Hence

$$|\mathbf{tr}((\tilde{\mathbf{\Pi}} - \mathbf{\Pi_0})\hat{\mathbf{B}}((\tilde{\mathbf{\Pi}} - \mathbf{\Pi_0})^{\mathbf{T}})\hat{\mathbf{A}})| \leq n \max_i |\langle \mathbf{Y_{i\cdot}}, \mathbf{X_{\cdot i}} \rangle| \leq n^2 L. \tag{17}$$

iv. Upper Bound of $\frac{||\mathbf{\Pi_0} - \tilde{\mathbf{\Pi}}||_F^2}{||\mathbf{\Pi_0}||_F^2}$.

From Inequalities (14), (15) and (17), we can obtain

$$\begin{aligned}&||(\mathbf{\Pi_0}-\tilde{\mathbf{\Pi}})||_F^2 \leq \frac{2}{nK}||(\mathbf{\Pi_0}-\tilde{\mathbf{\Pi}})\hat{\mathbf{B}}||_F^2\\ \leq&\frac{8}{nK}||\mathbf{\Pi_0}\hat{\mathbf{B}}\mathbf{\Pi_0^T}-\hat{\mathbf{A}}||_F^2+2\mathbf{tr}((\tilde{\mathbf{\Pi}}-\mathbf{\Pi_0})\hat{\mathbf{B}}((\tilde{\mathbf{\Pi}}-\mathbf{\Pi_0})^{\mathbf{T}})\hat{\mathbf{A}})\\ \leq&\frac{8}{nK}||\mathbf{\Pi_0}\hat{\mathbf{B}}\mathbf{\Pi_0^T}-\hat{\mathbf{A}}||_F^2+\frac{4nL}{K}.\end{aligned}$$

Since condition 2 holds, there exists a constant $\tilde{c}\geq 1$ such that $||\hat{\mathbf{A}}-\mathbf{\Pi_0}\hat{\mathbf{B}}\mathbf{\Pi_0}^{\mathbf{T}}||_F \leq \tilde{c}||\hat{\mathbf{A}}-\tilde{\mathbf{\Pi}}\hat{\mathbf{B}}\tilde{\mathbf{\Pi}}^{\mathbf{T}}||_F$. Therefore since $||\mathbf{\Pi_0}||_F^2=2n$ and the first and third condition, we can bound the relative NME when $n\to\infty$ as:

$$\begin{aligned}\frac{||(\mathbf{\Pi_0}-\tilde{\mathbf{\Pi}})||_F^2}{||\mathbf{\Pi_0}||_F^2} &\leq \frac{4}{n^2K}||\mathbf{\Pi_0}\hat{\mathbf{B}}\mathbf{\Pi_0^T}-\hat{\mathbf{A}}||_F^2+\frac{2L}{K}\\ &=\frac{4\tilde{c}}{n^2K}||\tilde{\mathbf{\Pi}}\hat{\mathbf{B}}\tilde{\mathbf{\Pi}}^{\mathbf{T}}-\hat{\mathbf{A}}||_F^2+\frac{2L}{K}\to 0.\end{aligned}$$

This completes our proof. □

Remark: Although Theorem 3 does not ensure NME $=0$ exactly, it makes sense in de-anonymization since NME can be neglected when the size of network is very large and we can map asymptotically all nodes correctly under mild conditions. We show the mildness of these conditions under a particularly network structure: the whole networks connected with high probability, which must follow $p_{\boldsymbol{C_iC_j}}=\Omega(\frac{\log n}{n}),\forall i,j\in\{1,2,...,n\}$ [13].

Meanwhile, we take $s=s_1=s_2=o(1)$ denoting sparse sampling from G. For condition (i), $\frac{L}{K}=O(\frac{p_{\boldsymbol{C_iC_j}}^2 s^2}{p_{\boldsymbol{C_iC_j}}s})=o(1)$; For conditions (ii) and (iii), $|\mathbf{E}[(\mathbf{A}-\mathbf{\Pi B\Pi^T})_{ij}]|=p_{\boldsymbol{C_iC_j}}s+p_{\boldsymbol{C_{\pi(i)}C_{\pi(j)}}}s=O(\frac{s\log n}{n})$ satisfies both; For condition (iv), we show in Section V-C that it holds if $\tilde{\mathbf{\Pi}}$ ensures that all nodes are mapped with those who have same community structures (Recall Lemma 1) as $\mathbf{\Pi_0}$, which is easily realizable.

B. Overlapping Communities Benefit De-anonymization

Intuitively, the large overlapping strength of the communities usually means that a large number of nodes belonging to different communities. In other words, the large overlapping strength enriches the feature of nodes in an average sense, whereby it could be beneficial to the de-anonymization process. Mathematically, a larger overlapping strength can lead to the increase of values in the weight matrix W according to its definition, thereby enlarges the differences between irrelevant nodes. Meanwhile, the overlapping communities can positively impact on reducing the relative NME through facilitating conditions in Theorem 3, specifically condition (iii). We now demonstrate this point in the following.

For convenience, we assume $s_1=s_2=s$. When π_0 maps all nodes to those who have same community structures, then on average condition (iii) in Theorem 3 can be written as

$$2\sum_{1\leq i<j\leq n} p_{\boldsymbol{C_iC_j}}s\log\left(\frac{1-p_{\boldsymbol{C_iC_j}}(2s-s^2)}{p_{\boldsymbol{C_iC_j}}(1-s)^2}\right)=o(Kn^2). \quad (18)$$

To characterize the global situation in the networks, we define an *average probability* $\hat{p}$ such that

$$\begin{aligned}&\sum_{1\leq i<j\leq n} p_{\boldsymbol{C_iC_j}}s\log\left(\frac{1-p_{\boldsymbol{C_iC_j}}(2s-s^2)}{p_{\boldsymbol{C_iC_j}}(1-s)^2}\right)\\ &=\frac{n(n-1)}{2}\log\left(\frac{1-\hat{p}(2s-s^2)}{\hat{p}(1-s)^2}\right)\hat{p}s,\end{aligned} \quad (19)$$

where $\hat{p}$ is positively correlated to the overlapping strength of the whole networks. Taking the derivative of $\hat{p}$ over $\left(\frac{1-\hat{p}(2s-s^2)}{\hat{p}(1-s)^2}\right)e^{\hat{p}s}$, we find that

$$d\left(\left(\frac{1-\hat{p}(2s-s^2)}{\hat{p}(1-s)^2}\right)e^{\hat{p}s}\right)/d\hat{p}=\frac{e^{\hat{p}s}}{(1-s)^2}(\frac{\hat{p}-1}{\hat{p}^2}-(2s-s^2))\leq 0, \quad (20)$$

indicating that $\left(\frac{1-\hat{p}(2s-s^2)}{\hat{p}(1-s)^2}\right)e^{\hat{p}s}$ is a decreasing function becoming smaller as overlapping strength increases. Therefore if the order of $\hat{p}$ rises, then the order of $||\hat{\mathbf{A}}-\mathbf{\Pi_0}\hat{\mathbf{B}}\mathbf{\Pi_0^T}||_F^2$ turns smaller, facilitating $||\hat{\mathbf{A}}-\mathbf{\Pi_0}\hat{\mathbf{B}}\mathbf{\Pi_0^T}||_F^2=o(Kn^2)$.

Taking a vivid example of the proposed OSBM [12] in which

$$p_{\boldsymbol{C_iC_j}}=\frac{1}{1+ae^{-x}}, \quad (21)$$

where a is an adjustable parameter and x is the number of overlapping communities. We find that $\min_{i,j}p_{\boldsymbol{C_iC_j}}=\frac{1}{1+a}$ is a constant if $a=\Omega(1)$, and can be arbitrarily close to 1 when x is large enough. So if $s=o(1)$ and $\hat{p}=1-o(1)$, which means that the overlapping strength is very large, then

$$\begin{aligned}\hat{p}\log(\frac{1-\hat{p}(2s-s^2)}{\hat{p}(1-s)^2})&=\hat{p}\log(1+\frac{1-\hat{p}}{\hat{p}(1-s)^2})\\ &\approx\frac{1-\hat{p}}{(1-s)^2}=o(\min_{i,j}p_{\boldsymbol{C_iC_j}}),\end{aligned} \quad (22)$$

thus condition (iii) holds. Meanwhile $s=o(1)$ makes condition (i) hold as well. Therefore all the four conditions in Theorem 3 hold, thus the relative NME vanishes to 0.

C. Complexity: WEMP can be Algorithmically Solved

Upon proving the good performance of solving WEMP in large-scale networks, now we algorithmically demonstrate that WEMP reduces the complexity of the MMSE problem since the optimal mapping of WEMP can be perfectly found by the convex-concave based de-anonymization algorithm (CBDA).

1) Formulation of WEMP in Constrained Optimization Form: We first restate WEMP in the form of the following constrained optimization problem:

$$\begin{aligned}&\text{minimize}\|(\hat{\mathbf{A}}-\mathbf{\Pi}\hat{\mathbf{B}}\mathbf{\Pi^T})\|_F^2\\ &\textbf{s.t.}\ \forall i\in V_1,\ \textstyle\sum_i\mathbf{\Pi}_{ij}=1 &(23)\\ &\forall j\in V_2,\ \textstyle\sum_j\mathbf{\Pi}_{ij}=1 &(24)\\ &\forall i,j,\ \mathbf{\Pi}_{ij}\in\{0,1\}, &(25)\\ &\forall i\in V_1, \boldsymbol{C_i}=\boldsymbol{C_{\pi(i)}} &(26)\end{aligned}$$

Constraints (23), (24) and (25) are the attributes of permutation matrices. What's more, we append constraint (26), which means that our estimated mapping π should keep the community representation of all the nodes in V_1 unchanged before and after mapping. Note that it is hard to implement this constraint directly in the optimization problem since it is not in the form of permutation matrix. However, we

can easily convert it into a suitable one by defining a new matrix to characterize the community representation of all the nodes, which we call as "Community Representation Matrix", denoted as $\mathbf{M}$. Its formal definition is as follows.

Take Fig. 1 as an instance again. The community representation matrix of G, denoted as $\mathbf{M_G}$, satisfies

$$\mathbf{M_G^T} = \begin{bmatrix} 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 \end{bmatrix}.$$

Note that the community representation matrices for G, G_1 and G_2 are identical. So we set all of them to be $\mathbf{M}$. Hence the constraint (26) can be rewritten as $||\mathbf{\Pi M} - \mathbf{M}||_F^2 = 0$. According to optimization theory, we can form this constraint into the objective function by regarding it as the penalty term and obtain a new objective function

$$F_0(\mathbf{\Pi}) = ||\hat{\mathbf{A}} - \mathbf{\Pi}\hat{\mathbf{B}}\mathbf{\Pi^T}||_{\mathrm{F}}^2 + \mu||\mathbf{\Pi M} - \mathbf{M}||_F^2,$$

where μ is an adjustable penalty parameter, which is large enough such that when the objective function reaches its minimum value, $||\mathbf{\Pi M} - \mathbf{M}||_F^2$ is exactly. Note that this transformation of objective function does not affect the previous analytical results of WEMP since we have the assumption that the true mapping ensures that all nodes are mapped with those who have same community structures.

2) Problem Relaxation and Idea of Algorithm Design: Hereinafter, we focus on how we design our algorithm targeting the WEMP.

Problem Relaxation: WEMP is an integer program problem which cannot be solved efficiently. We relax the original feasible region of WEMP Ω_0 into Ω, which are respectively

$$\Omega_0 = \{\mathbf{\Pi}_{ij} \in \{0,1\} | \forall i,j, \textstyle\sum_i \mathbf{\Pi}_{ij} = 1 \ , \sum_j \mathbf{\Pi}_{ij} = 1\};$$
$$\Omega = \{\mathbf{\Pi}_{ij} \in [0,1] | \forall i,j, \textstyle\sum_i \mathbf{\Pi}_{ij} = 1 \ , \sum_j \mathbf{\Pi}_{ij} = 1\}.$$

After this relaxation the problem becomes tractable. However, a natural question arises: *How to obtain the solution of the original unrelaxed problem from that of the relaxed problem*?

Idea of Convex-Concave Relaxation Method: Note that the minimizer of a concave function must be at the boundary of the feasible region, coinciding that Ω_0, the original feasible set, is just the boundary of Ω. Therefore, a natural idea emerges: *We can modify the convex relaxed problem into a concave problem gradually.* Thus we apply the convex-concave optimization method (CCOM), whose concept is pioneeringly proposed in [35] to solve pattern matching problems: For $F_0(\mathbf{\Pi})$, we find its convex and concave relaxed version respectively $F_1(\mathbf{\Pi})$ and $F_2(\mathbf{\Pi})$. Then we obtain a new objective function as $F(\mathbf{\Pi}) = (1-\alpha)F_1(\mathbf{\Pi}) + \alpha F_2(\mathbf{\Pi})$. We modify α gradually from 0 to 1 with interval $\Delta\alpha$, each time solving the new $F(\mathbf{\Pi})$ initialized by the optimizer last time. $F(\mathbf{\Pi})$ becomes more concave, with its optimum closer to Ω_0 where $\tilde{\mathbf{\Pi}}$ lies.

*3) **Implementation of CCOM and Algorithm Design**:* Although [35] has proposed the general framework of CCOM, the way it presents to obtain $F_1(\mathbf{\Pi})$ and $F_2(\mathbf{\Pi})$ is rather complex, as it involves Kronecker product and the Laplacian matrix of graphs. Here we provide a simple way, as defined in Lemma 2, to get the convex relaxation and concave relaxation.

Lemma 2. *A proper way to get the convex relaxation and concave relaxation is*

$$F_1(\mathbf{\Pi}) = F_0(\mathbf{\Pi}) + \frac{\lambda_{min}}{2}(n - ||\mathbf{\Pi}||_F^2);$$
$$F_2(\mathbf{\Pi}) = F_0(\mathbf{\Pi}) + \frac{\lambda_{max}}{2}(n - ||\mathbf{\Pi}||_F^2).$$

where λ_{min} (λ_{max}) is the smallest (largest) eigenvalue of the Hessian matrix of $F_0(\mathbf{\Pi})$. Therefore we form our new objective function in CCOM as

$$F(\mathbf{\Pi}) = (1-\alpha)F_1(\mathbf{\Pi}) + \alpha F_2(\mathbf{\Pi}) = F_0(\mathbf{\Pi}) + 2\xi(n - ||\mathbf{\Pi}||_F^2),$$

where $\xi = (1-\alpha)\lambda_{min} + \alpha\lambda_{max}$, $\xi \in [\lambda_{min}, \lambda_{max}]$.

The proof of Lemma 2, which is left in Appendix E, uses the sufficient and necessary condition that for a function whose variable is matrix is convex (concave) is that the Hessian matrix of this function is positive (negative) semi-definite.

Lemma 2 presents a simple way to implement CCOM algorithmically, since $F_0(\mathbf{\Pi})$ is just our objective function in Section V-C1 and $||\mathbf{\Pi}||_F^2$ can be computed easily. We can modify $F(\mathbf{\Pi})$ step by step from a convex function to a concave function by modifying the value of ξ or α. In the following analysis, we set $F_\xi(\mathbf{\Pi})$ equivalent to $F(\mathbf{\Pi})$ since ξ is an adjustable parameter in $F(\mathbf{\Pi})$.

A vivid example of the CCOM under the formulation of $F_\xi(\mathbf{\Pi})$ by Lemma 2 is illustrated in Fig. 2. As can be seen in the figure, when ξ starts at λ_{min}, $F_\xi(\mathbf{\Pi})$ is a convex function, thus we can obtain the minimizer of this objective function. After we find the minimizer, we increase α which makes the objective function become less convex. To obtain the minimizer of this new objective function, we have the prior knowledge of the previous minimizer, and since we only slightly modify the objective function, the optimal solution of new objective function should not deviate much from the previous one intuitively. Therefore we can start from the previous minimizer to find the new minimizer. Gradually, as α becomes increasingly larger, the objective function tends to be concave while the minimizer of it tends to get close to the boundary, on which the optimal solution of the original WEMP exists. The trail for the minimizer can be referred to the red line with arrows in Fig. 2.

Based on the above analysis, we propose Algorithm 1,

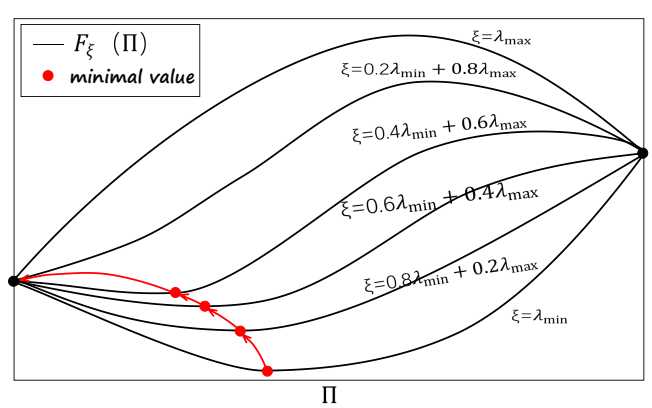


Fig. 2: An Illustration of the Implementation of CCOM.

Algorithm 1 Convex-concave Based De-anonymization Algorithm (CBDA)

Input: Adjacent matrices $\mathbf{A}$ and $\mathbf{B}$; Community assignment matrix $\mathbf{M}$; Weight controlling parameter μ; Adjustable parameters δ, $\Delta\xi$.

Output: Estimated permutation matrix $\tilde{\mathbf{\Pi}}$.

1: Form the objective function $F_0(\mathbf{\Pi})$ and $F(\mathbf{\Pi})$.
2: $\xi \leftarrow 0$, $k \leftarrow 1$. $\mathbf{\Pi_1} \leftarrow \mathbf{1}_{n\times n}./n$. Set ξ_m, the upper limit of ξ.
3: **while** $\xi < \xi_m$ and $\mathbf{\Pi_k} \notin \Omega_0$ **do**
4: **while** $k = 1$ or $|F(\mathbf{\Pi_{k+1}}) - F(\mathbf{\Pi_k})| \geq \delta$ **do**
5: $\mathbf{X}^{\perp} \leftarrow \arg\min_{\mathbf{X}^{\perp}} \mathbf{tr}(\nabla_{\mathbf{\Pi_k}} F(\mathbf{\Pi_k})^T \mathbf{X}^{\perp})$, where $\mathbf{X}^{\perp} \in \Omega$.
6: $\gamma_k \leftarrow \arg\min_{\gamma} F(\mathbf{\Pi_k} + \gamma(\mathbf{X}^{\perp} - \mathbf{\Pi_k}))$, where $\gamma_k \in [0, 1]$.
7: $\mathbf{\Pi_{k+1}} \leftarrow \mathbf{\Pi_k} + \gamma_k(\mathbf{X}^{\perp} - \mathbf{\Pi_k})$, $k \leftarrow k + 1$.
8: **end while**
9: $\xi \leftarrow \xi + \Delta\xi$.
10: **end while**
11: $\tilde{\mathbf{\Pi}} = \mathbf{\Pi_k}$.

called *Convex-concave Based De-anonymization Algorithm (CBDA)*, as our main algorithm for the weighted-edge matching problem (WEMP) under CCOM. Note that $F_0(\mathbf{\Pi})$ itself is convex in our problem, thus we can set ξ from 0 to an arbitrarily large number, which obviates the great complexity to calculate eigenvalues of Hessian matrices.

CBDA consists of an outer loop (lines 3 to 10) and an inner loop (lines 4 to 8). The outer loop modifies ξ in CCOM. The inner loop finds the minimizer of $F(\mathbf{\Pi})$, whose main idea resembles descending algorithms: In line 5, we obtain descending direction by minimizing $\mathbf{tr}(\nabla_{\mathbf{\Pi_k}} F(\mathbf{\Pi_k})^T \mathbf{X}^{\perp})$, dangling the highest probability to find a descending direction characterized by $\mathbf{tr}(\nabla_{\mathbf{\Pi_k}} F(\mathbf{\Pi_k})^T \mathbf{X}^{\perp}) < 0$. In line 6 we search for step length γ_k contributing most to lowering $F(\mathbf{\Pi})$ on this descending direction. Line 7 is the update of estimation.

4) Time Complexity and Convergence Analysis: **Time Complexity:** The inner loop is similar to the Frank-Wolfe algorithm, with $O(n^6)$ in a round (since the input is an $n \times n$ matrix). If the maximum number of inner loops as T, thus the whole algorithm has a complexity of $O\big(\frac{n^6 T\xi}{\Delta\xi}\big)$. As far as we know, a dearth of algorithmic analysis of seedless de-anonymization exists except for [36], [37], with their proposed algorithm sharing identical complexity of $O(n^6)$ with ours.

Convergence: Before the convergence analysis, we first clarify that:

- We set $\mathbf{\Pi_k}$ as the estimation after k rounds in the inner loop, thus we have $\mathbf{\Pi_{k+1}} = \mathbf{\Pi_k} + \gamma_k(\mathbf{X}_{\perp} - \mathbf{\Pi_k})$.
- We set $F_{\xi}(\mathbf{\Pi}) = F_0(\mathbf{\Pi}) + \xi(n - ||\mathbf{\Pi}||_F^2)$ and $\mathbf{\Pi}^{\xi}$ as the minimizer of $F_{\xi}(\mathbf{\Pi})$.

Then we analyze the convergence of CBDA and propose Lemma 3.

Lemma 3. *CBDA converges and its final output is a permutation matrix within the original feasible region Ω_0.*

Proof. There are inner and outer loops in CBDA and we show the convergence of them respectively.

1. Inner Loop: We provide the outline of the proof here and put the detailed one in Appendix F.

We focus on $F_{\xi}(\mathbf{\Pi_{k+1}})$ and $F_{\xi}(\mathbf{\Pi}^{\xi})$. According to Taylor's Theorem, we can derive that

$$\begin{aligned} F_{\xi}(\mathbf{\Pi_{k+1}}) &= F_{\xi}(\mathbf{\Pi_k} + \gamma_k(\mathbf{X}_{\perp} - \mathbf{\Pi_k})) \\ &\leq F_{\xi}(\mathbf{\Pi_k}) + \gamma_k \mathbf{tr}(\nabla F_{\xi}^T(\mathbf{\Pi_k})(\mathbf{\Pi}^{\xi} - \mathbf{\Pi_k})) + \gamma_k \mathbf{R_k}, \end{aligned} \tag{27}$$

$$\begin{aligned} F_{\xi}(\mathbf{\Pi}^{\xi}) &= F_{\xi}(\mathbf{\Pi_k} + \mathbf{\Pi}^{\xi} - \mathbf{\Pi_k}) \\ &= F_{\xi}(\mathbf{\Pi_k}) + \mathbf{tr}(\nabla F_{\xi}^T(\mathbf{\Pi_k})(\mathbf{\Pi}^{\xi} - \mathbf{\Pi_k})) + \mathbf{R'_k}, \end{aligned} \tag{28}$$

where $\gamma_k \mathbf{R_k}$ and $\mathbf{R'_k}$ is the remainder of this Taylor series.

Combining Eqn. (27) and (28), we can obtain

$$\begin{aligned} &F_{\xi}(\mathbf{\Pi_{k+1}}) - F_{\xi}(\mathbf{\Pi}^{\xi}) \\ &\leq (1 - \gamma_k)(F_{\xi}(\mathbf{\Pi_k}) - F_{\xi}(\mathbf{\Pi}^{\xi})) + \gamma_k \Delta\mathbf{R_k} \\ &\leq \prod_{i=1}^{k}(1 - \gamma_i)(F_{\xi}(\mathbf{\Pi_1}) - F_{\xi}(\mathbf{\Pi}^{\xi})) + \sum_{i=1}^{k} \gamma_i \prod_{j=1}^{k-i}(1 - \gamma_j)\Delta\mathbf{R}_i. \end{aligned} \tag{29}$$

For $F_{\xi}(\mathbf{\Pi_1}) - F_{\xi}(\mathbf{\Pi}^{\xi})$, note that $\mathbf{\Pi_1} = \mathbf{\Pi}^{\xi - \mathbf{\Delta}\xi}$, then we can derive that

$$F_{\xi}(\mathbf{\Pi}^{\xi - \mathbf{\Delta}\xi}) - F_{\xi}(\mathbf{\Pi}^{\xi}) \leq \Delta\xi(||\mathbf{\Pi}^{\xi - \mathbf{\Delta}\xi}||_F^2 - ||\mathbf{\Pi}^{\xi}||_F^2). \tag{30}$$

Therefore by combining Inequalities (29) and (30), we can obtain that both $\prod_{i=1}^{k}(1 - \gamma_i)(F_{\xi}(\mathbf{\Pi_1}) - F_{\xi}(\mathbf{\Pi}^{\xi}))$ and $\sum_{i=1}^{k} \gamma_i \prod_{j=1}^{k-i}(1 - \gamma_j)\Delta\mathbf{R}_i$ approach to 0.

2. Outer Loop: Note that from Eqn. (30), we know $(||\mathbf{\Pi}^{\xi - \mathbf{\Delta}\xi}||_F^2 - ||\mathbf{\Pi}^{\xi}||_F^2)$ is nonnegative since $\Delta\xi > 0$ and $\mathbf{\Pi}^{\xi}$ is the minimizer of $F_{\xi}(\mathbf{\Pi})$. Thus $||\mathbf{\Pi}^{\xi}||_F^2 \leq ||\mathbf{\Pi}^{\xi - \mathbf{\Delta}\xi}||_F^2$. Note that $||\mathbf{\Pi}||_F^2 \leq n$. From Inequality (30), we find that

$$\begin{aligned} F_{\xi}(\mathbf{\Pi}^{\xi}) &\geq F_0(\mathbf{\Pi}^{\xi - \mathbf{\Delta}\xi}) + (\xi - \Delta\xi)(n - ||\mathbf{\Pi}^{\xi - \mathbf{\Delta}\xi}||_F^2) \\ &\quad - \Delta\xi(n - ||\mathbf{\Pi}^{\xi}||_F^2) \\ &= F_{\xi - \Delta\xi}(\mathbf{\Pi}^{\xi - \mathbf{\Delta}\xi}) - \Delta\xi \mathbf{tr}(||\mathbf{\Pi}^{\xi}||_F^2 - n). \end{aligned}$$

Therefore

$$\begin{aligned} &|F_{\xi}(\mathbf{\Pi}^{\xi}) - F_{\xi - \Delta\xi}(\mathbf{\Pi}^{\xi - \mathbf{\Delta}\xi})| \\ &\leq \Delta\xi|||(\mathbf{\Pi}^{\xi})||_F^2 - n| \leq \Delta\xi|||\mathbf{\Pi}^{\xi - \mathbf{\Delta}\xi}||_F^2 - n| \\ &\leq \Delta\xi|||\mathbf{\Pi}^{\xi_0}||_F^2 - n| \leq \Delta\xi(n - 1), \end{aligned}$$

where the third inequality holds since $\mathbf{\Pi}^{\xi_0}$ is the minimizer of $F_{\lambda_{min}}(\mathbf{\Pi})$, i.e., the convex relaxation of $F_0(\mathbf{\Pi})$, and the fourth inequality holds since $\min_{\mathbf{\Pi}\in\Omega} ||\mathbf{\Pi}||_F^2 = 1$ and $\mathbf{\Pi} = \mathbf{1}_{n\times n}./n$ is the minimizer. Therefore, if $\Delta\xi = o\big(\frac{1}{n}\big)$, the outer loop converges.

Combining the convergence analysis of both inner and outer loops above, we complete the proof of the convergence of CBDA. □

Lemma 3 ensures that CBDA can perfectly solve WEMP, which vanishes the relative NME. In this section, we have proved that the CBDA algorithm, proposed as the algorithmic approach for seedless de-anonymization, is feasible for networks with broad degree distributions and can bring high accuracy in mapping those networks in its feasible region.

5) ***Discuss on the Computational Complexity of CBDA:*** Different from the literature focusing on seeded de-anonymization problem, our analysis and algorithm are pro-

posed in the seedless case. Although the seeded case can iteratively and locally de-anonymize the nodes with the aid of a fraction of pre-identified nodes, the seedless case can only be solved with a global perspective, which inevitably results in a high complexity of $\Omega(n^3)$. As far as we know, most seedless works face the same bottleneck that their polynomial-time algorithms cannot directly apply to large-scale networks due to the high time complexity. For instance, [26] realizes a "fast" de-anonymization algorithm with time costs $O(n^2d^2)$, where d is the maximal degree of nodes in the networks. For a dense network, their complexity approaches $O(n^4)$. Besides, the path following algorithm in [27] costs $O(n^7)$ in the worst case. [36], which de-anonymizes the networks with non-overlapping communities with an MAP estimator, also costs $O(n^6)$ to obtain their solutions. Even the most intuitive heuristic named fast approximate quadratic (FAQ) programming [28], which has no analytical guarantee and has a relatively low complexity of $O(n^3)$, performs their experiments with hundreds of or thousands of nodes.

To sum up, though the complexity of algorithms may decrease by adopting some heuristics, the performance may also decrease. Meanwhile, the design of efficient algorithms for large-scale seedless de-anonymization still remains open.

VI. Experimental Aspect of Social Network De-anonymization Problem

In this section, we utilize three datasets: synthetic networks, sampled real social networks and true cross-domain networks, to validate our theoretical results and the performance of CBDA.Before presenting empirical results, we first introduce our experimental setup.

A. Experiment Setup

1) Main Parameters: We list our adjustable parameters involved in our experiments in Table II. Three parameters are in need of further explanations:

i) a. This is a parameter in the overlapping stochastic block model (OSBM) which determines the $p_{\boldsymbol{C_iC_j}}$, the probability of edge existence between nodes i and j in underlying graph. Specifically, $p_{\boldsymbol{C_iC_j}}$ can be expressed as

$$p_{\boldsymbol{C_iC_j}} = \frac{1}{1+ae^{-x}}, \tag{31}$$

where x is the number of communities that both nodes i and j belong to. Note that if a becomes larger (smaller), then $p_{\boldsymbol{C_iC_j}}$ is smaller (larger) so that the graph becomes sparser (denser).

ii) η. This is the community ratio. It means the ratio between the number of communities and nodes. This ratio reflects the density of community numbers regardless of the size of network. In performance validation of CBDA we set $\eta = 0.05$ or 0.1, while when studying the influence of η on de-anonymization accuracy, it will be endowed with more values.

iii) OL/NOL. OL means that communities are overlapping while NOL means not. This makes for illustrating the impact of the overlapping property of communities on the mapping accuracy.

TABLE II: **Main Experimental Parameters**

Notation	Definition	Range
N	Number of Nodes	{500, 1000, 1500, 2000}
s	Sampling Probability ($s_1 = s_2 = s$)	0.3-0.9
a	OSBM Parameter	{3, 5, 7, 9}
η	Community Ratio	{0.05, 0.1}
OL/NOL	Overlapping or Non-Overlapping	{OL, NOL}

TABLE III: **Datasets in Basic Experiments**

Dataset	Synthetic	Sampled Real Social	Cross-Domain Co-author
Source	OSBM	LiveJournal [14]	MAG [15]
Num. of Nodes	$500 \sim 2000$	$500 \sim 2000$	3176
Num. of Communities	$25 \sim 1000$	$25 \sim 1000$	89

2) Experimental Datasets: We discuss three adopted datasets, which is shown is Table III, in an order from model-based to real social networks.

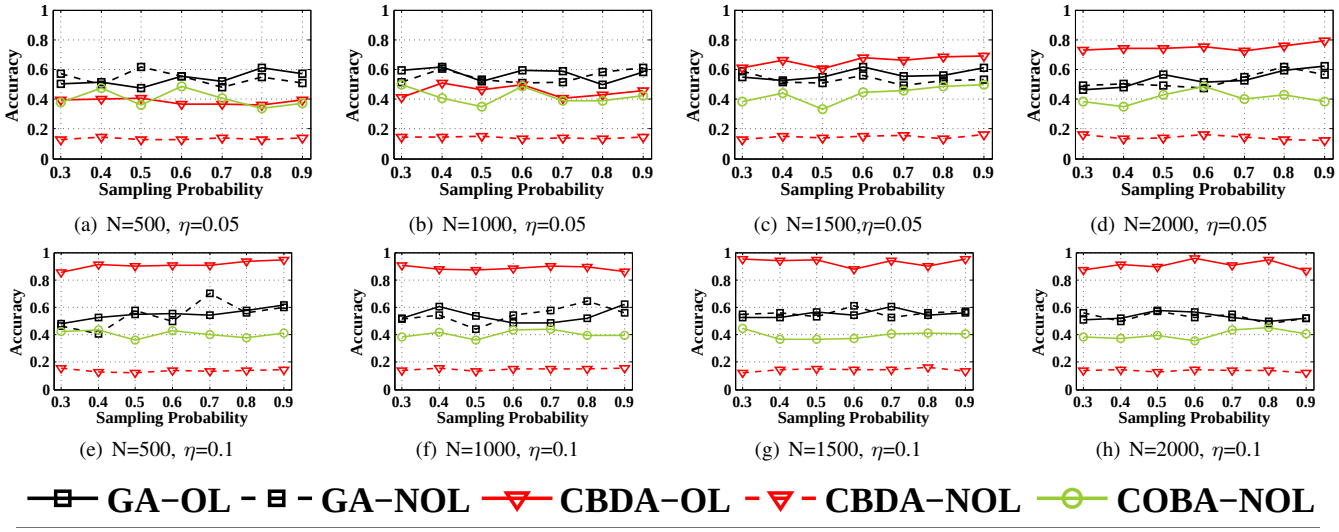
i) Synthetic Networks: We generate networks by setting the community representation of every node independently and randomly deciding the edge existence in node pair (i, j) based on OSBM [12]

ii) Sampled Real Social Networks: The underlying social network G is extracted from LiveJournal [29], while G_1 and G_2 are sampled from G with the same probability s.

iii) Cross-Domain Co-author Networks: The co-author networks are from the Microsoft Academic Graph (MAG) [15]. We extract 4 networks belonging to different sub-areas in the field of computer science, with the same group of authors, each of whom has a unique 8-bit hexadecimal ID enabling us to construct the true mapping between two networks as the one mapping nodes with same ID. Each network can be viewed as G_1 or G_2, thus there are $C_4^2 = 6$ combinations. Note that we can assign w_{ij} on all these 3 datasets since the prior knowledge is just the community assignment matrix M, which can be generated or known from the real networks.

3) Algorithms for Comparison and Performance Metric: We exclude algorithms for seeded de-anonymization and select algorithms suitable for seedless cases related to our main point: showing the impact of overlapping communities on reducing NME, though other algorithms might outperform ours. We select two algorithms for comparison: (i) the Genetic Algorithm (GA), an epitome of heuristic algorithms which can work under both overlapping or non-overlapping communities; (ii) the Convex Optimization-Based Algorithm (COBA) in [36], [37], which primarily suits the non-overlapping cases by assigning every node to a unique community. The performance metric in our experiments is the *accuracy*, i.e., the proportion of correctly mapped nodes.

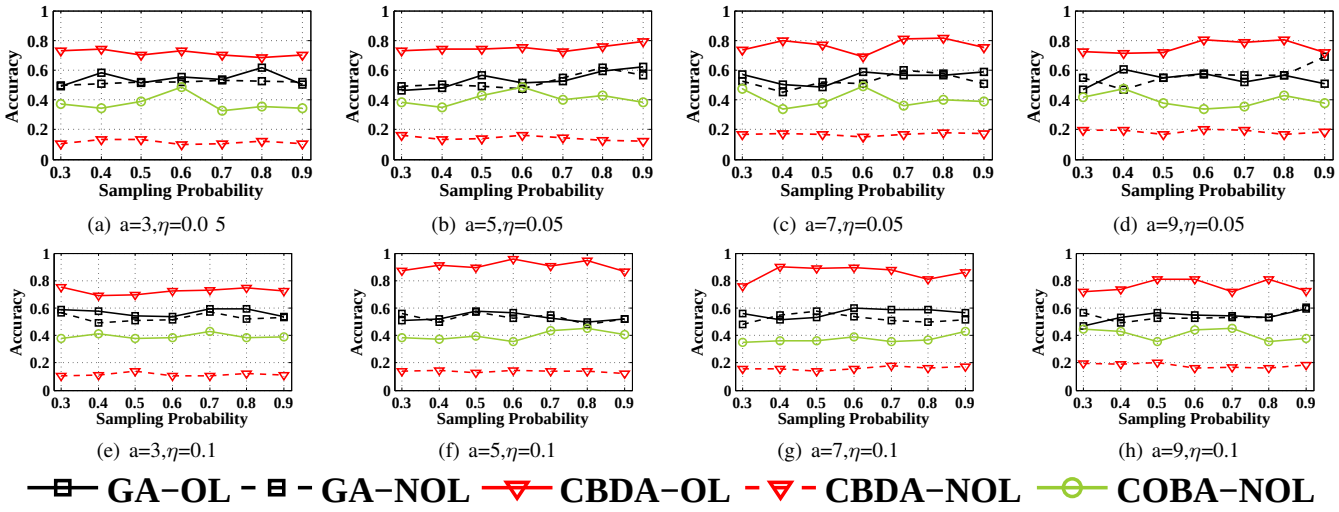
4) Supplementary Experiments: To make our experimental validation more comprehensive and convincing, we study (i) the effect of different community ratios (η), which is modified from 0.025 to 0.2 with interval 0.025, on the accuracy based on sampled real social networks; (ii) the priority of our cost function with $\mathbf{W}$ derived from MMSE makes for higher accuracy, comparing with the cost function without $\mathbf{W}$ in [6]; (iii) the instability of GA revealing its practical limitation, for which we take the average performance of 10 duplicate experiments as the final accuracy.



(a) N=500, η=0.05 (b) N=1000, η=0.05 (c) N=1500,η=0.05 (d) N=2000, η=0.05

(e) N=500, η=0.1 (f) N=1000, η=0.1 (g) N=1500, η=0.1 (h) N=2000, η=0.1

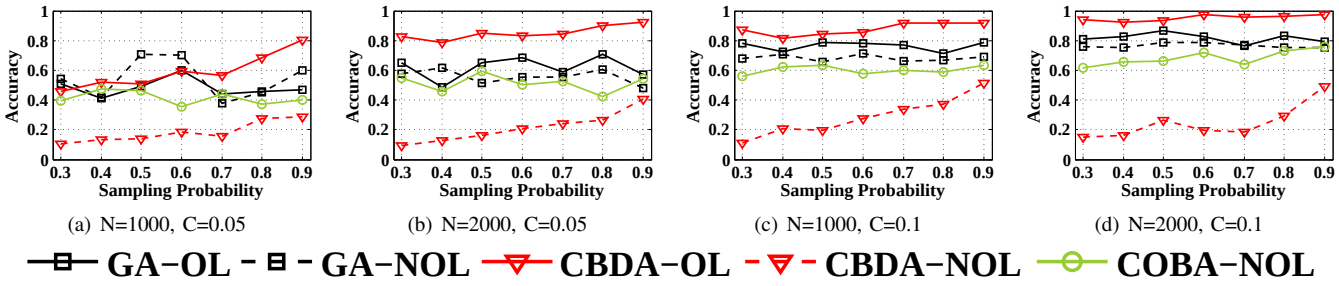
Fig. 3: Experiments on Synthetic Networks with $a = 5$.



(a) a=3,η=0.0 5 (b) a=5,η=0.05 (c) a=7,η=0.05 (d) a=9,η=0.05

(e) a=3,η=0.1 (f) a=5,η=0.1 (g) a=7,η=0.1 (h) a=9,η=0.1

Fig. 4: Experiments on Synthetic Networks with $N = 2000$.



(a) N=1000, C=0.05 (b) N=2000, C=0.05 (c) N=1000, C=0.1 (d) N=2000, C=0.1

Fig. 5: Experiments on Sampled Real Social Networks.

B. Experiment Results

1) Synthetic Networks: Figs. 3 and 4 illustrate our experimental results on synthetic networks, where the community ratio $\eta \in \{0.05, 0.1\}$, the network size N range from 500 to 2000 in Fig 3 and the OSBM parameter a range from 3 to 9 in Fig 4. From Fig. 3, we observe that: (i) The *average* accuracy of genetic algorithm (GA) under different settings keeps at levels around $40\% - 60\%$, which illustrates that different sizes, densities and whether the communities overlap or not do not make many differences on the performance of GA averagely. This is because GA examines the edges one by one to make the cost function as small as possible, therefore it is not seriously affected by the global setting of the networks. (ii) The accuracy of COBA also keeps at a stable level in different situations. However, COBA can only cope with non-overlapping situations, and generally its performance is inferior to GA when communities are not overlapped, which is in line with the results in [36], [37]. (iii) The accuracy of

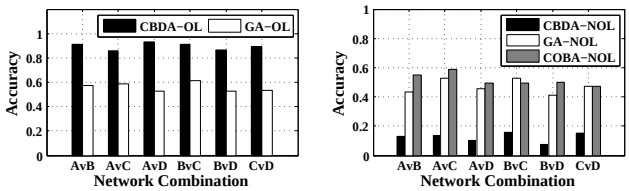
CBDA, our algorithm, rises with the increasing network size N when $\eta = 0.05$. Specifically, when N goes from 500 to 2000, the accuracy rises from approximately 40% to 80%. This corresponds to our Theorem 3 that as the size of networks becomes larger, the relative NME becomes smaller. When $\eta = 0.1$, which indicates denser communities, the accuracy of CBDA is at a high level (around 90%) even if the network size is small. On the other hand, however, when dealing with non-overlapping situations, our CBDA works stably but not as efficiently as GA or COBA, with the accuracy only around 20%. This indicates that the overlapping strength brings great differences to our algorithm.

From Fig. 4, we can observe that when communities are non-overlapping, for both COBA and our CBDA, curves at $\eta = 0.05$ have the same trends as curves at $\eta = 0.1$, showing that the community density under non-overlapping situations does not affect the performance of all these algorithms. However, our CBDA always performs better than other algorithms when the communities are overlapping each other. With a certain community size, the accuracy of CBDA keeps at a stable level when $\eta = 0.05$ and varies with edge density a when $\eta = 0.1$. Thus, when the community density is large, the performance of CBDA is mainly decided by the edge density (a), positively correlated to community density; when the community density is small, then the performance of CBDA is mainly decided by the size of the networks (N). This shows that the community ratio (density) determines the dominant factor (a or N) in de-anonymization accuracy in networks with overlapping communities.

2) Sampled Real Social Networks: In sampled real social networks, we utilize the real underlying network, thus no modifications on a exist. The results are in Fig. 5. We can observe: (i) GA performs better in larger networks and under denser communities, either overlapping or non-overlapping; (ii) The performance of COBA is also enhanced when the size of networks become larger and the community becomes denser; (iii) The performance of CBDA under non-overlapping situations does not outperform other algorithms, but a rising tendency exists as the sampling probability s becomes larger; (iv) The performance of CBDA under overlapping situations still performs well under denser communities and larger network size, with the highest point 95% and the highest average level around 90% when $N = 2000$ and $\eta = 0.1$, the largest size and densest communities in Table III.

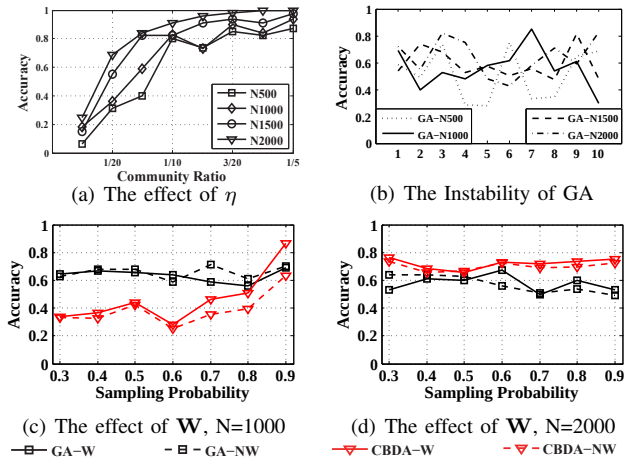
Synthesizing the above four observations, we can learn that the OSBM does not reflect the real social networks very precisely, since the performance of all three algorithms under non-overlapping or overlapping communities differs in two datasets. Moreover, with the same experimental setting, we discover that the performance of our CBDA is better in sampled real social networks than in OSBM-based synthetic networks, which further undergirds the high performance of our algorithm in practical use. Additionally, the results in Fig. 5 also meet Theorem 3 that as the network size becomes larger, the relative NME is much smaller and close to 0, indicating that Theorem 3 also works in real social networks.

3) Cross-Domain Co-author Networks: In cross-domain co-author networks, we pick up four networks with the same set of 3176 users. Fig. 6 illustrates our results. We find that in non-overlapping situation, the results correspond to those in previous datasets that our CBDA does not perform well, while GA and COBA work well. On the other hand, in overlapping situation, we find our CBDA reaches accuracy around 90%, outstripping GA whose accuracy is averagely 60%. This phenomenon upgrades the significance of our CBDA in de-anonymization with overlapping communities since the dataset is entirely realistic. Moreover, since overlapping cases are much more quotidian in real social networks, CBDA has wider usage than GA and COBA.



(a) Overlapping (b) Non-Overlapping

Fig. 6: Experiments on Cross-Domain Co-author Networks.



(a) The effect of η (b) The Instability of GA

(c) The effect of $\mathbf{W}$, N=1000 (d) The effect of $\mathbf{W}$, N=2000

Fig. 7: Supplementary Experiments

4) The Effect of Community Density: We apply the sampled real social networks under which we can adjust the community ratio η. We modify η from 0.025 to 0.2, with interval 0.025. The results are shown in Fig. 7(a). We can observe that our CBDA performs better when the network size is larger, which again echoes the conclusion in Theorem 3. Moreover, with the larger community ratio, the accuracy of CBDA rises up, showing that CBDA is suitable for social networks with highly overlapping communities. What's more, a huge gap occurs between the accuracy of $\eta = 0.025$ and $\eta = 0.075$, and when $\eta \geq 0.1$, the accuracy of CBDA under all the network sizes involved keeps at high levels, around 80% or higher. The results further illustrate that the higher community ratio η, the better de-anonymizing result will be.

5) The Effect of Weight Matrix $\mathbf{W}$*:* As Figs. 7(c) and 7(c) show, CBDA works better appending $\mathbf{W}$ derived by MMSE, since the non-weighted cost function, adopted in [6], fails to distinguish nodes belonging to different number of communities. It shows the superiority of cost functions derived with rationale, as we claim in Section IV. Under larger network size, however, the difference becomes fainter since the impact of distinguishing a single node by w_{ij} is weaker than

the benefits brought by large size shown in Theorem 3.

6) The Instability of Genetic Algorithm: We disclose the instability of GA in Fig. 7(b). We run GA 10 times under sampled real social networks with different sizes. The performance of GA fluctuates violently, bewildering adversaries in the quality of a specific estimation, which inhibits the usage of GA in practice.

C. Discussion for the Experimental Results

In general cases, our CBDA algorithm outperforms others in the experiments, because this algorithm can properly cope with the overlapping community structure in the social networks, while other works are either a simple heuristic without analytical guarantee or just proposed under non-overlapping communities. Every node is divided into a specific community in the non-overlapping community case while a node may belong to different communities in our background setting. Though the Genetic algorithm is unstable for the de-anonymization work, the COBA algorithm in [36], [37] is expected to achieve similar performance if this algorithm can also be properly adapted to make use of overlapping communities. Due to the difficulty in doing so, we leave it a future work after we can achieve new progress in the de-anonymization problem.

The performance of CBDA is poor in non-overlapping community case. This phenomenon stems in our analysis in Section V-B, which requires that the overlapping strength of the communities should be large to satisfy our optimal mapping conditions proposed in Theorem 3.

VII. Conclusion

We tackle the seedless de-anonymization problem under a more practical social network model concreted by *overlapping communities* than existing works. With the MMSE, we derive a well-justified cost function, i.e., the MMSE estimator, which aims to minimize the expected number of mismatched users. While showing the NP-hardness of minimizing the proposed MMSE estimator, we validly transform it into the WEMP problem, which resolves the tension between the optimality and the complexity: (i) WEMP can be algorithmically solved via CBDA, which is proved to find exactly the optimum of WEMP; (ii) The solution of WEMP is proved to asymptotically achieve optimal with a negligible mapping error under mild conditions facilitated by higher overlapping strength. Extensive experiments further confirm the effectiveness of CBDA under overlapping communities.

Acknowledgments

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Luoyi Fu received the B.E. degree in electronic engineering and the Ph.D. degree in computer science and engineering from Shanghai Jiao Tong University, China, in 2009 and 2015, respectively. She is currently an Assistant Professor with the Department of Computer Science and Engineering, Shanghai Jiao Tong University. Her research of interests is in the area of social networking and big data, connectivity analysis, and random graphs. She has been a member of the Technical Program Committees of several conferences, including ACM MobiHoc 2018-2019 and IEEE INFOCOM 2018-2019.

Jiapeng Zhang received the B.E. degree in the Department of Electronic Engineering and Information Science from the University of Science and Technology of China, Hefei, China, in 2017. He is now pursuing a Ph.D. degree in Electronic Engineering at Shanghai Jiao Tong University, China. His research interests include random graph theory and social network de-anonymization.

Xinyu Wu received the B.E. degree from the Department of Electrical Engineering, Shanghai Jiao Tong University, China, in 2018. He is currently pursuing the Ph.D. degree with the Massachusetts Institute of Technology, Massachusetts, MA, USA. He was a Research Intern supervised by Dr. L. Fu. His research interests include wireless networks, social networks, and network privacy.

Xinbing Wang received the B.S. degree (Hons.) in automation from Shanghai Jiao Tong University, Shanghai, China, in 1998, the M.S. degree from Computer Science and Technology, Tsinghua University, Beijing, China, in 2001, and the Ph.D. degree major in electrical and computer engineering and minor in mathematics from North Carolina State University, Raleigh, NC, USA, in 2006. He is currently a Professor with the Department of Electronic Engineering, Shanghai Jiaotong University. He was a member of the Technical Program Committees of ACM MobiCom 2012, 2018-2019, ACM MobiHoc 2012-2014, and IEEE INFOCOM 2009-2017. He has been an Associate Editor of the IEEE/ACM Transactions on Networking and the IEEE Transactions on Mobile Computing.

Guihui Chen received the B.S. degree from Nanjing University, the M.E. degree from Southeast University, and the Ph.D. degree from The University of Hong Kong. He visited the Kyushu Institute of Technology, Japan, in 1998, as a Research Fellow, and the University of Queensland, Australia, in 2000, as a Visiting Professor. From 2001 to 2003, he was a Visiting Professor with Wayne State University. He is currently a Distinguished Professor and a Deputy Chair with the Department of Computer Science, Shanghai Jiao Tong University. His research interests include wireless sensor networks, peer-to-peer computing, and performance evaluation. He is a member of the IEEE Computer Society. He has served on technical program committees of numerous international conferences.

APPENDIX A
PROOF OF THEOREM 1

Proof. Define $\mathcal{G}_{\mathbf{\Pi}}$ as the set of all possible realizations of the underlying network G which can be sampled to form the observed networks G_1, G_2 given the mapping matrix $\mathbf{\Pi}$. Then the MMSE estimator can be written as

$$\hat{\mathbf{\Pi}} = \arg\min_{\mathbf{\Pi}\in\mathbf{\Pi^n}} \sum_{\mathbf{\Pi_0}\in\Pi^n} ||\mathbf{\Pi}-\mathbf{\Pi_0}||_F^2 \sum_{G\in\mathcal{G}_{\mathbf{\Pi}}} Pr(G,\mathbf{\Pi_0}|G_1,G_2,\boldsymbol{\theta}).$$

Let us focus on the conditional probability $Pr(G,\mathbf{\Pi_0}|G_1,G_2,\boldsymbol{\theta})$ in Eqn. (2). According to Bayesian's formula, along with the fact that G_1 and G_2 are sampled independently from each other, we obtain

$$\begin{aligned} Pr(G,\mathbf{\Pi_0}|G_1,G_2,\boldsymbol{\theta}) &= \frac{Pr(G,G_1,G_2,\mathbf{\Pi_0})}{Pr(G_1,G_2)} \\ &\sim Pr(G)Pr(G_1|G)Pr(G_2|G,\mathbf{\Pi_0}), \end{aligned} \tag{32}$$

where $a\sim b$ means that a and b are positively correlated and their difference appears only in the parameters unrelated to $\mathbf{\Pi_0}$, which will not change the value of $\arg\max$ or $\arg\min$.[3] Note that the parameter set $\boldsymbol{\theta}$ remains invariant, so we need not add $\boldsymbol{C_i}$ and $\boldsymbol{\theta}$ into further consideration.

Set E^{ij} as the indicator variable about whether an edge exists between nodes i and j in the edge set E. If an edge exists then $E^{ij}=1$, otherwise $E^{ij}=0$. The same rule also holds for indicators E_1^{ij} and E_2^{ij}. Therefore Eqn. (32) can be further written as

$$\begin{aligned} &\sum_{G\in\mathcal{G}_{\mathbf{\Pi}}} Pr(G)Pr(G_1|G)Pr(G_2|G,\Pi_0) \\ =&\sum_{G\in\mathcal{G}_{\mathbf{\Pi}}}\prod_{i<j}^{n} s_1^{E_1^{ij}}(1-s_1)^{E^{ij}-E_1^{ij}} s_2^{E_2^{\pi_0(i)\pi_0(j)}} \\ &\cdot(1-s_2)^{E^{ij}-E_2^{\pi_0(i)\pi_0(j)}} p_{\boldsymbol{C_iC_j}}^{E^{ij}}(1-p_{\boldsymbol{C_iC_j}})^{1-E^{ij}} \\ =&\prod_{i<j}\left(\frac{s_1}{1-s_1}\right)^{E_1^{ij}}\left(\frac{s_2}{1-s_2}\right)^{E_2^{\pi_0(i)\pi_0(j)}} \\ &\cdot\sum_{G\in\mathcal{G}_{\mathbf{\Pi}}}\left((1-s_1)(1-s_2)\frac{p_{\boldsymbol{C_iC_j}}}{1-p_{\boldsymbol{C_iC_j}}}\right)^{E^{ij}} \\ \sim&\sum_{G\in\mathcal{G}_{\mathbf{\Pi}}}\left((1-s_1)(1-s_2)\frac{p_{\boldsymbol{C_iC_j}}}{1-p_{\boldsymbol{C_iC_j}}}\right)^{E^{ij}}. \end{aligned} \tag{33}$$

Note that the last equivalence in Eqn. (33) holds since the term $\left(\frac{s_1}{1-s_1}\right)^{E_1^{ij}}$ does not depend on π_0 and the product $\prod_{i<j}\left(\frac{s_2}{1-s_2}\right)^{E_2^{\pi_0(i)\pi_0(j)}}$ is independent of π_0 due to the bijective property of π_0.

Then we define $G_{\pi_0}^*$ as the graph which has the smallest number of edges in $\mathcal{G}_{\mathbf{\Pi}}$. Equivalently $G_{\pi_0}^*=(V,E_1\cup\pi_0(E_1))$, where $\pi_0(E_1)=\{(\pi_0(i),\pi_0(j))|(i,j)\in E_1\}$. Now we set $E_{\pi_0}^*$ as the edge set of $G_{\pi_0}^*$, and $E_{\pi_0}^{*ij}$ as the indicator variable between nodes i and j, i.e., $E_{\pi_0}^{*ij}=1$ if $(i,j)\in E_{\pi_0}^*$ and $E_{\pi_0}^{*ij}=0$ otherwise. Then we sum up all the graphs in $\mathcal{G}_{\mathbf{\Pi}}$

$$\begin{aligned} &\sum_{G\in\mathcal{G}_{\mathbf{\Pi}}}\left((1-s_1)(1-s_2)\frac{p_{\boldsymbol{C_iC_j}}}{1-p_{\boldsymbol{C_iC_j}}}\right)^{E^{ij}} \\ =&\prod_{i<j}^{n}\left((1-s_1)(1-s_2)\frac{p_{\boldsymbol{C_iC_j}}}{1-p_{\boldsymbol{C_iC_j}}}\right)^{E_{\pi_0}^{*ij}} \\ &\cdot\sum_{k=0}^{E_{ij}-E_{\pi_0}^{*ij}} C_{E_{ij}-E_{\pi_0}^{*ij}}^{k}\left((1-s_1)(1-s_2)\frac{p_{\boldsymbol{C_iC_j}}}{1-p_{\boldsymbol{C_iC_j}}}\right)^{k}. \end{aligned} \tag{34}$$

Note that in Eqn. (34) last multiplicative factor ,

$$\sum_{k=0}^{E_{ij}-E_{\pi_0}^{*ij}} C_{E_{ij}-E_{\pi_0}^{*ij}}^{k}\left((1-s_1)(1-s_2)\frac{p_{\boldsymbol{C_iC_j}}}{1-p_{\boldsymbol{C_iC_j}}}\right)^{k},$$

yields as a Bernoulli sum, therefore Eqn. (34) can be further written as

$$\begin{aligned} &\sum_{G\in\mathcal{G}_{\mathbf{\Pi}}}\left((1-s_1)(1-s_2)\frac{p_{\boldsymbol{C_iC_j}}}{1-p_{\boldsymbol{C_iC_j}}}\right)^{E^{ij}} \\ =&\prod_{i<j}^{n}\left((1-s_1)(1-s_2)\frac{p_{\boldsymbol{C_iC_j}}}{1-p_{\boldsymbol{C_iC_j}}}\right)^{E_{\pi_0}^{*ij}} \\ &\cdot\left(1+(1-s_1)(1-s_2)\frac{p_{\boldsymbol{C_iC_j}}}{1-p_{\boldsymbol{C_iC_j}}}\right)^{1-E_{\pi_0}^{*ij}} \\ \sim&\prod_{i<j}^{n}\left(\frac{p_{\boldsymbol{C_iC_j}}(1-s_1)(1-s_2)}{1-p_{\boldsymbol{C_iC_j}}(s_1+s_2-s_1s_2)}\right)^{E_{\pi_0}^{*ij}} \\ \sim&\sum_{i<j}^{n}E_{\pi_0}^{*ij}\log\left(\frac{p_{\boldsymbol{C_iC_j}}(1-s_1)(1-s_2)}{1-p_{\boldsymbol{C_iC_j}}(s_1+s_2-s_1s_2)}\right). \end{aligned} \tag{35}$$

Here the last line in Eqn. (35) holds since the log operator keeps the minimum $\mathbf{\Pi_0}$ invariant. Note that $G_{\pi_0}^*=(V,E_1\cup\pi_0(E_1))$. Then we can find that $E_{\mathbf{\Pi_0}}^{*ij}=0$ if and only if both E_1^{ij} and E_2^{ij} are equal to 0, and $E_{\mathbf{\Pi_0}}^{*ij}=1$ occurs in the following three conditions:

- $(i,j)\in E_1$ but $(i,j)\notin E_2$. Note that this condition also ensures that $(\pi_0(i),\pi_0(j))\in E_2$.
- $(i,j)\in E_2$ but $(i,j)\notin E_1$. Note that this condition also ensures that $(\pi_0(i),\pi_0(j))\notin E_2$.
- $(i,j)\in E_1$ and $(i,j)\in E_2$. Note that this condition also ensures that $(\pi_0(i),\pi_0(j))\in E_2$.

Synthesizing all the above conditions, we express $E_{\pi_0}^{*ij}$ as

$$E_{\pi_0}^{*ij}=\frac{1}{2}(E_1^{ij}+E_2^{ij}+|\mathbb{1}\{(i,j)\in E_1\}-\mathbb{1}\{(\pi_0(i),\pi_0(j))\in E_2\}|), \tag{36}$$

where $\mathbb{1}\{P\}=1$ if the random event P happens and $\mathbb{1}\{P\}=0$ otherwise. Substituting Eqn. (36) into the last line in Eqn. (35), we get

$$\begin{aligned} &\arg\min_{\mathbf{\Pi}\in\mathbf{\Pi^n}}\sum_{i<j}^{n}E_{\pi_0}^{*ij}\log\left(\frac{p_{\boldsymbol{C_iC_j}(1-s_1)(1-s_2)}}{1-p_{\boldsymbol{C_iC_j}}(s_1+s_2-s_1s_2)}\right) \\ =&\arg\max_{\mathbf{\Pi}\in\mathbf{\Pi^n}}\sum_{i<j}^{n}w_{ij}|\mathbb{1}\{(i,j)\in E_1\}-\mathbb{1}\{(\pi_0(i),\pi_0(j))\in E_2\}| \\ =&\arg\max_{\mathbf{\Pi}\in\mathbf{\Pi^n}}||\mathbf{W}\circ(\mathbf{\Pi_0A}-\mathbf{B\Pi_0})||_F^2, \end{aligned} \tag{37}$$

where $w_{ij}=\log\left(\frac{1-p_{\boldsymbol{C_iC_j}}(s_1+s_2-s_1s_2)}{p_{\boldsymbol{C_iC_j}}(1-s_1)(1-s_2)}\right)$ is weight between

[3]There is a notation abuse for $\sim$ between the one in Eqn. (6) and here.

i and j, $\mathbf{W}$ is a symmetric weight matrix where $\mathbf{W}(i,j) = \sqrt{w_{ij}} = \mathbf{W}(j,i)$, and "$\circ$" denotes the Hadamard product.

Substituting Eqn. (37) into Eqn. (32), now we can formulate the MMSE estimator as

$$\hat{\mathbf{\Pi}} = \arg\max_{\mathbf{\Pi}\in\mathbf{\Pi^n}} \sum_{\mathbf{\Pi_0}\in\mathbf{\Pi^n}} ||\mathbf{\Pi}-\mathbf{\Pi_0}||_F^2 ||\mathbf{W}\circ(\mathbf{\Pi_0}\mathbf{A}-\mathbf{B}\mathbf{\Pi_0})||_F^2. \quad (38)$$

□

Appendix B
Analysis of the Transformation from MMSE to WEMP

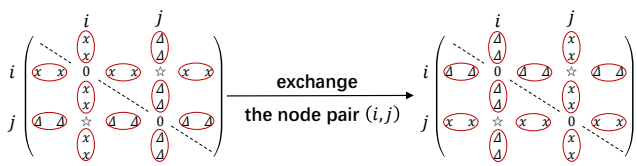


Fig. 8: An example of the effect of $\mathbf{\Pi_0}$ which differs from $\tilde{\mathbf{\Pi_0}}$ only in the i_{th} and j_{th} row. The triangles denote the j_{th} row and column the "x"es denote the i_{th} row and column of $\mathbf{W}\circ(\mathbf{\Pi_0}\mathbf{A}\mathbf{\Pi_0^T}-\mathbf{B})$. And the triangles denote the i_{th} row and column the "x"es denote the j_{th} row and column of $\mathbf{W}\circ(\tilde{\mathbf{\Pi}}\mathbf{A}\tilde{\mathbf{\Pi}}^{\mathbf{T}}-\mathbf{B})$. Note that the difference between $\mathbf{W}\circ(\mathbf{\Pi_0}\mathbf{A}\mathbf{\Pi_0^T}-\mathbf{B})$ and $\mathbf{W}\circ(\tilde{\mathbf{\Pi}}\mathbf{A}\tilde{\mathbf{\Pi}}^{\mathbf{T}}-\mathbf{B})$ exists in the i_{th} and j_{th} row and column except the intersections (those 0s and stars).

i. Analysis of $||\mathbf{\Pi_0}\hat{\mathbf{A}}-\hat{\mathbf{B}}\mathbf{\Pi_0}||_F^2$ where $\mathbf{\Pi_0}\in S_2(\mathbf{\Pi})$.

Now we focus on the value of $||\mathbf{\Pi_0}\hat{\mathbf{A}}-\hat{\mathbf{B}}\mathbf{\Pi_0}||_F^2$, where $\mathbf{\Pi_0}\in S_2(\mathbf{\Pi})$. Note that any permutation in $S_2(\mathbf{\Pi})$ only causes matching error on one pair of nodes. Thus if we consider $\mathbf{\Pi}=\tilde{\mathbf{\Pi}}$ and set one specific $\mathbf{\Pi_0}\in S_2(\tilde{\mathbf{\Pi}})$, which differs from $\tilde{\mathbf{\Pi}}$ only in the i_{th} and j_{th} row, we can derive that

$$\begin{aligned}
&||\mathbf{\Pi_0}\hat{\mathbf{A}}-\hat{\mathbf{B}}\mathbf{\Pi_0}||_F^2 - ||\tilde{\mathbf{\Pi}}\hat{\mathbf{A}}-\hat{\mathbf{B}}\tilde{\mathbf{\Pi}}||_F^2\\
=&||\mathbf{W}\circ(\mathbf{\Pi_0}\mathbf{A}\mathbf{\Pi_0^T}-\mathbf{B})||_F^2 - ||\mathbf{W}\circ(\tilde{\mathbf{\Pi}}\mathbf{A}\tilde{\mathbf{\Pi}}^{\mathbf{T}}-\mathbf{B})||_F^2\\
=&2\Big(\sum_{k\neq i,j}^{n}[(\mathbf{W}\circ(\mathbf{\Pi_0}\mathbf{A}\mathbf{\Pi_0^T}-\mathbf{B}))_{ik}^2-(\mathbf{W}\circ(\tilde{\mathbf{\Pi}}\mathbf{A}\tilde{\mathbf{\Pi}}^{\mathbf{T}}-\mathbf{B}))_{ik}^2]\\
&+\sum_{k\neq i,j}^{n}[(\mathbf{W}\circ(\mathbf{\Pi_0}\mathbf{A}\mathbf{\Pi_0^T}-\mathbf{B}))_{jk}^2-(\mathbf{W}\circ(\tilde{\mathbf{\Pi}}\mathbf{A}\tilde{\mathbf{\Pi}}^{\mathbf{T}}-\mathbf{B}))_{jk}^2]\Big)\\
=&2\Big(\sum_{k\neq i,j}^{n}w_{ik}[(\mathbf{\Pi_0}\mathbf{A}\mathbf{\Pi_0^T}-\mathbf{B})_{ik}^2-(\tilde{\mathbf{\Pi}}\mathbf{A}\tilde{\mathbf{\Pi}}^{\mathbf{T}}-\mathbf{B})_{ik}^2]\\
&+\sum_{k\neq i,j}^{n}w_{jk}[(\mathbf{\Pi_0}\mathbf{A}\mathbf{\Pi_0^T}-\mathbf{B})_{jk}^2-(\tilde{\mathbf{\Pi}}\mathbf{A}\tilde{\mathbf{\Pi}}^{\mathbf{T}}-\mathbf{B})_{jk}^2]\Big)\\
=&2\Big(\sum_{k\neq i,j}^{n}w_{ik}[\mathbf{\Pi_0}\mathbf{A}\mathbf{\Pi_0^T}-\tilde{\mathbf{\Pi}}\mathbf{A}\tilde{\mathbf{\Pi}}^{\mathbf{T}}]_{ik}\psi(\mathbf{B}_{ik})\\
&+\sum_{k\neq i,j}^{n}w_{jk}[\mathbf{\Pi_0}\mathbf{A}\mathbf{\Pi_0^T}-\tilde{\mathbf{\Pi}}\mathbf{A}\tilde{\mathbf{\Pi}}^{\mathbf{T}}]_{jk}\psi(\mathbf{B}_{jk})\Big),
\end{aligned} \quad (39)$$

where $\psi(x)=-1$ if $x=1$ and $\psi(x)=1$ if $x=0$. Fig. 8 illustrates how Eqn. (39) can be derived intuitively. Note that if $\mathbf{\Pi_0}$ and $\tilde{\mathbf{\Pi}}$ are different only in the i_{th} and j_{th} rows, then the difference between $||\mathbf{W}\circ(\mathbf{\Pi_0}\mathbf{A}\mathbf{\Pi_0^T}-\mathbf{B})||_F^2$ and $||\mathbf{W}\circ(\tilde{\mathbf{\Pi}}\mathbf{A}\tilde{\mathbf{\Pi}}^{\mathbf{T}}-\mathbf{B})||_F^2$ exists in the red circles in Fig. 8, which corresponds to the third line in Eqn. (39). Note that the intersection part, i.e., the stars in Fig. 8, does not contribute to the $||\mathbf{W}\circ(\mathbf{\Pi_0}\mathbf{A}\mathbf{\Pi_0^T}-\mathbf{B})||_F^2$ and $||\mathbf{W}\circ(\tilde{\mathbf{\Pi}}\mathbf{A}\tilde{\mathbf{\Pi}}^{\mathbf{T}}-\mathbf{B})||_F^2$.

Note that since $\mathbf{\Pi_0}$ and $\tilde{\mathbf{\Pi}}$ are different in the i_{th} and j_{th} rows, then $(\tilde{\mathbf{\Pi}}\mathbf{A}\tilde{\mathbf{\Pi}}^{\mathbf{T}})_{ik}=(\mathbf{\Pi_0}\mathbf{A}\mathbf{\Pi_0^T})_{jk}$. Therefore

$$\begin{aligned}
&||\mathbf{\Pi_0}\hat{\mathbf{A}}-\hat{\mathbf{B}}\mathbf{\Pi_0}||_F^2 - ||\tilde{\mathbf{\Pi}}\hat{\mathbf{A}}-\hat{\mathbf{B}}\tilde{\mathbf{\Pi}}||_F^2\\
=&2\Big(\sum_{k\neq i,j}^{n}w_{ik}\psi(\mathbf{B}_{ik})([\mathbf{\Pi_0}\mathbf{A}\mathbf{\Pi_0^T}]_{ik}-[\mathbf{\Pi_0}\mathbf{A}\mathbf{\Pi_0^T}]_{jk})\\
&+\sum_{k\neq i,j}^{n}w_{jk}\psi(\mathbf{B}_{jk})([\mathbf{\Pi_0}\mathbf{A}\mathbf{\Pi_0^T}]_{jk}-[\mathbf{\Pi_0}\mathbf{A}\mathbf{\Pi_0^T}]_{ik})\Big)\\
=&2\Big(\sum_{k\neq i,j}^{n}(w_{ik}\psi(\mathbf{B}_{ik})-w_{jk}\psi(\mathbf{B}_{jk}))\\
&\cdot[(\mathbf{\Pi_0}\mathbf{A}\mathbf{\Pi_0^T})_{ki}-(\mathbf{\Pi_0}\mathbf{A}\mathbf{\Pi_0^T})_{kj}]\Big).
\end{aligned} \quad (40)$$

In convenience, we define the term in this equation as $\Delta_{i,j,\pi} = (w_{ik}\psi(\mathbf{B}_{ik})-w_{jk}\psi(\mathbf{B}_{jk}))\cdot[(\mathbf{\Pi_0}\mathbf{A}\mathbf{\Pi_0^T})_{ki}-(\mathbf{\Pi_0}\mathbf{A}\mathbf{\Pi_0^T})_{kj}]$. Since G_1 and G_2 are independently sampled from G, their adjacency matrices $\mathbf{A}$ and $\mathbf{B}$ should be conditionally independent, which follows $\mathbf{E}[\mathbf{A}_{\pi_0(i)\pi_0(j)}\mathbf{B}_{ij}|\mathbf{U}_{ij}]=\mathbf{E}[\mathbf{A}_{\pi_0(i)\pi_0(j)}|\mathbf{U}_{ij}]\mathbf{E}[\mathbf{B}_{ij}|\mathbf{U}_{ij}]$, where $\mathbf{U}$ is the adjacency matrix of network G. Thereafter, we can get the conditionally expectation of $\psi(\mathbf{B}_{ij})$ as

$$\begin{aligned}
&\mathbf{E}[\psi(\mathbf{B}_{ij})|\mathbf{U}_{ij}=1]=-1\cdot s_2+1\cdot(1-s_2)=1-2s_2,\\
&\mathbf{E}[\psi(\mathbf{B}_{ij})|\mathbf{U}_{ij}=0]=1.
\end{aligned}$$

In the sequel, we can estimate the expectation of $\Delta_{i,j,k}$ on matrix $\mathbf{A},\mathbf{B}$ by dividing it into four conditions as follows.

$$\begin{aligned}
&\mathbf{E}_{\mathbf{A},\mathbf{B}}[\Delta_{i,j,\pi}]\\
=&\mathbf{E}[\Delta_{i,j,\pi}|\mathbf{U}_{ik}=1,\mathbf{U}_{jk}=1]Pr(\mathbf{U}_{ik}=1,\mathbf{U}_{jk}=1)\\
&+\mathbf{E}[\Delta_{i,j,\pi}|\mathbf{U}_{ik}=0,\mathbf{U}_{jk}=0]Pr(\mathbf{U}_{ik}=0,\mathbf{U}_{jk}=0)\\
&+\mathbf{E}[\Delta_{i,j,\pi}|\mathbf{U}_{ik}=1,\mathbf{U}_{jk}=0]Pr(\mathbf{U}_{ik}=1,\mathbf{U}_{jk}=0)\\
&+\mathbf{E}[\Delta_{i,j,\pi}|\mathbf{U}_{ik}=1,\mathbf{U}_{jk}=0]Pr(\mathbf{U}_{ik}=1,\mathbf{U}_{jk}=0)\\
=&(w_{ik}-w_{jk})(1-2s_2)(s_1-s_1)\cdot p_{\boldsymbol{C_iC_k}}p_{\boldsymbol{C_jC_k}}\\
&+(w_{ik}-w_{jk})(0-0)\cdot(1-p_{\boldsymbol{C_iC_k}})(1-p_{\boldsymbol{C_jC_k}})\\
&+(w_{ik}(1-2s_2)-w_{jk})(s_1-0)\cdot p_{\boldsymbol{C_iC_k}}(1-p_{\boldsymbol{C_jC_k}})\\
&+(w_{ik}-w_{jk}(1-2s_2))(0-s_1)\cdot(1-p_{\boldsymbol{C_iC_k}})p_{\boldsymbol{C_jC_k}}\\
=&(w_{ik}-2w_{ik}s_2-w_{jk})s_1p_{\boldsymbol{C_iC_k}}-(w_{ik}-w_{jk}+2w_{jk}s_2)s_1p_{\boldsymbol{C_jC_k}}\\
&+2(w_{ik}+w_{jk})s_1s_2p_{\boldsymbol{C_iC_k}}p_{\boldsymbol{C_jC_k}}.
\end{aligned}$$

In this equation, Δ_{i,j,k,π_0} reflects a part of the difference $||\mathbf{\Pi_0}\hat{\mathbf{A}}-\hat{\mathbf{B}}\mathbf{\Pi_0}||_F^2-||\tilde{\mathbf{\Pi}}\hat{\mathbf{A}}-\hat{\mathbf{B}}\tilde{\mathbf{\Pi}}||_F^2$ caused by the difference of a single element in matrices $\mathbf{\Pi_0}\hat{\mathbf{A}}-\hat{\mathbf{B}}\mathbf{\Pi_0}$ and $\tilde{\mathbf{\Pi}}\hat{\mathbf{A}}-\hat{\mathbf{B}}\tilde{\mathbf{\Pi}}$.[4] Since we consider the average case of all possible $\mathbf{\Pi_0}$, we also consider the average value of Δ_{i,j,π_0}, which we denote as $\hat{\Delta}=\mathbf{E}_{i,j,\pi_0}[\Delta_{i,j,\pi_0}]$. Note that $\mathbf{E}_{\mathbf{A},\mathbf{B}}[||\mathbf{\Pi_0}\hat{\mathbf{A}}-\hat{\mathbf{B}}\mathbf{\Pi_0}||_F^2-||\tilde{\mathbf{\Pi}}\hat{\mathbf{A}}-\hat{\mathbf{B}}\tilde{\mathbf{\Pi}}||_F^2]>0$ since $\tilde{\mathbf{\Pi}}$ is the minimizer of $||\mathbf{\Pi_0}\hat{\mathbf{A}}-\hat{\mathbf{B}}\mathbf{\Pi_0}||_F^2$. Therefore $\hat{\Delta}=\mathbf{E}_{i,j,\pi_0}[\Delta_{i,j,\pi_0}]>0$.

ii. Analysis of $\sum_{\mathbf{\Pi_0}\in S_k(\mathbf{\Pi})}||\mathbf{\Pi_0}\hat{\mathbf{A}}-\hat{\mathbf{B}}\mathbf{\Pi_0}||_F^2$.

Now we move to the second part involved in our idea. We first focus on $S_k(\mathbf{\Pi_0})$, and count the number of elements in $S_k(\mathbf{\Pi_0})$, denoted as $|S_k|$. Note that if there are k mismatched nodes in a graph with n nodes, there are C_n^k possible sets of mismatched nodes. We define $|T_k|$ as the number of elements in each possible set, and can get $|S_k|=C_n^k|T_k|$. For $|T_k|$, we

[4]For example, the difference of the corresponding element (with the same notation, e.g., (i,k) in the left matrix and (j,k) in the right matrix, both of which are triangles.) in two matrices in Fig. 8 inside one of the red circles

can find that it satisfies

$$\begin{aligned}|T_k| &= (k-1)(|T_{k-2}| + (k-2)(|T_{k-3}| + (k-3)(|T_{k-4}| + ...))) \\ &= \sum_{t=1}^{k-1}(\prod_{i=1}^{t}(k-i))|T_{k-t-1}|.\end{aligned} \tag{41}$$

Consider $|T_k|$ and $|T_{k-1}|$ in Eqn. (41), we can discover that

$$|T_k| = (k-1)(|T_{k-2}| + |T_{k-1}|) \geq (k-1)|T_{k-1}|, k \geq 2.$$

Therefore we obtain the relationship between $|S_k|$ and $|S_{k-1}|$ as

$$|S_k| = C_n^k|T_k| \geq (k-1)\frac{C_n^k}{C_n^{k-1}}|S_{k-1}| = (1-\frac{1}{k})(n-k+1)|S_{k-1}|, \tag{42}$$

where $k \geq 2$. Eqn. (42) shows that when k is much smaller than n, then $\frac{|S_k|}{|S_{k-1}|} = (1-\frac{1}{k})(n-k+1)$ is large; when k gets close to n, then $\frac{|S_k|}{|S_{k-1}|}$ approaches 1, which means that $|S_k|$ and $|S_{k-1}|$ are almost the same.

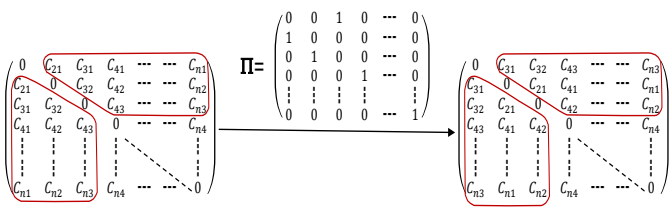


Fig. 9: An example of the effect of $\mathbf{\Pi} \in S_3(\tilde{\mathbf{\Pi}})$, where we set $\tilde{\mathbf{\Pi}} = \mathbf{I}$. $\mathbf{I}$ is the identity matrix. Note that under the $\mathbf{\Pi}$ above the arrow, which differs from $\mathbf{I}$ only in the first three rows (columns). Thus the possible difference between two matrices only exists in the red circles, with $6n-6$ elements in the matrix involved.

Now we consider $\mathbf{\Pi_0} \in S_k(\mathbf{\Pi})$. Note that for any $\mathbf{\Pi_0} \in S_k(\mathbf{\Pi})$, there are k rows and columns that may cause the difference between $||\mathbf{W} \circ (\mathbf{\Pi_0 A \Pi_0^T} - \mathbf{B})||_F^2$ and $||\mathbf{W} \circ (\tilde{\mathbf{\Pi}} \mathbf{A} \tilde{\mathbf{\Pi}}^{\mathbf{T}} - \mathbf{B})||_F^2$. Fig. 9 illustrates an example of $\mathbf{\Pi_0} \in S_3(\mathbf{\Pi})$. Therefore we can discover for any $\mathbf{\Pi_0} \in S_k(\mathbf{\Pi})$, the number of node pairs (i,j) which may influence the difference between $||\mathbf{W} \circ (\mathbf{\Pi_0 A \Pi_0^T} - \mathbf{B})||_F^2$ and $||\mathbf{W} \circ (\tilde{\mathbf{\Pi}} \mathbf{A} \tilde{\mathbf{\Pi}}^{\mathbf{T}} - \mathbf{B})||_F^2$ is approximately $\sum_{i=1}(n-i) = \frac{(2n-k-1)k}{2}$.[5] Thus, denoting N_k as this number of node pair, we can obtain

$$\begin{aligned}N_k &= \frac{(2n-k-1)k}{2}|S_k| \\ &\geq \frac{(2n-k-1)k}{2}(1-\frac{1}{k})(n-k+1)|S_{k-1}| \\ &= (1-\frac{1}{k})(n-k+1)\frac{(2n-k-1)k}{(2n-k)(k-1)}N_{k-1} \\ &= (n-k+1)\frac{2n-k-1}{2n-k}N_{k-1}.\end{aligned}$$

[5]For example, in Fig. 9 when $k=3$ the number is $6n-6$. Although there may be some elements which do not cause error, such as the two stars in Fig. 8, the number of this kinds of node pairs can be neglected when n is large enough.

Therefore in average, we have

$$\begin{aligned}&\sum_{\mathbf{\Pi_0} \in S_k} ||\mathbf{\Pi_0 \hat{A}} - \mathbf{\hat{B} \Pi_0}||_F^2 = N_k \hat{\Delta} \\ &\geq (n-k+1)\frac{2n-k-1}{2n-k}N_{k-1}\hat{\Delta} \\ &\geq (n-k+1)\frac{2n-k-1}{2n-k}\sum_{\mathbf{\Pi_0} \in S_{k-1}} ||\mathbf{\Pi_0 \hat{A}} - \mathbf{\hat{B} \Pi_0}||_F^2 \\ &\approx (n-k+1)\sum_{\mathbf{\Pi_0} \in S_{k-1}} ||\mathbf{\Pi_0 \hat{A}} - \mathbf{\hat{B} \Pi_0}||_F^2,\end{aligned} \tag{43}$$

where the last approximation holds since $k \leq n$ and when $n \to \infty$, $\frac{2n-k-1}{2n-k} \to 1$.

Therefore, we can claim that in average, if $k_1 > k_2$, then

$$\sum_{\mathbf{\Pi_0} \in S_{k_1}} ||\mathbf{\Pi_0 \hat{A}} - \mathbf{\hat{B} \Pi_0}||_F^2 > \sum_{\mathbf{\Pi_0} \in S_{k_2}} ||\mathbf{\Pi_0 \hat{A}} - \mathbf{\hat{B} \Pi_0}||_F^2. \tag{44}$$

iii. Maximum Value Under Sequence Inequality.

In average case, by setting $\mathbf{\Pi}$ in the original MMSE objective function

$$\sum_{\mathbf{\Pi_0} \in \Pi^n} ||\mathbf{\Pi} - \mathbf{\Pi_0}||_F^2 ||\mathbf{W} \circ (\mathbf{\Pi_0 A} - \mathbf{B \Pi_0})||_F^2$$

equal to $\tilde{\mathbf{\Pi}}$, the minimizer of WEMP, then this original MMSE objective function reaches its largest value under Sequence Inequality.

Moreover, note that if we do not set $\tilde{\mathbf{\Pi}} = \hat{\mathbf{\Pi}}$, for example set $\tilde{\mathbf{\Pi}} = \mathbf{\Pi} \in S_k(\mathbf{\Pi})$, we can verify that $\mathbf{\Pi}$ does not make the objective function in Eqn. (10) larger than $\mathbf{\Pi_0}$ since

$$\begin{aligned}&0||\mathbf{\Pi \hat{A} \Pi^T} - \mathbf{\hat{B}}||_{\mathbf{F}}^{\mathbf{2}} + 2k||\mathbf{\Pi \hat{A} \Pi^T} - \mathbf{\hat{B}}||_{\mathbf{F}}^{\mathbf{2}} \\ &\geq 2k||\mathbf{\hat{\Pi} \hat{A} \hat{\Pi}^T} - \mathbf{\hat{B}}||_{\mathbf{F}}^{\mathbf{2}} + 0||\mathbf{\Pi \hat{A} \Pi^T} - \mathbf{\hat{B}}||_{\mathbf{F}}^{\mathbf{2}},\end{aligned}$$

which means that the Sequency Inequality preserves that when $||\mathbf{\Pi_0 \hat{A} \Pi_0^T} - \mathbf{\hat{B}}||_F^2$ achieves its minimum, then $||\Pi - \Pi_0||_F^2$ also achieves its minimum. Therefore by setting $\tilde{\mathbf{\Pi}} = \hat{\mathbf{\Pi}}$ we can achieve the largest value of the original MMSE problem under this sequence inequality.

Appendix C
Proof of Theorem 2

Proof.

$$\begin{aligned}&g(\hat{\mathbf{\Pi}}) - g(\tilde{\mathbf{\Pi}}) \\ &= \sum_{\mathbf{\Pi_0} \in \mathbf{\Pi^n}} (||\hat{\mathbf{\Pi}} - \mathbf{\Pi_0}||_F^2 - ||\tilde{\mathbf{\Pi}} - \mathbf{\Pi_0}||_F^2)||\mathbf{\Pi_0 \hat{A}} - \mathbf{\hat{B} \Pi_0}||_F^2.\end{aligned} \tag{45}$$

Then we divide the set $\mathbf{\Pi^n}$ into two subsets:

$$\begin{aligned}\mathbf{\Pi_1^n} &= \{\mathbf{\Pi} \in \mathbf{\Pi^n} | ||\hat{\mathbf{\Pi}} - \mathbf{\Pi_0}||_F^2 > ||\tilde{\mathbf{\Pi}} - \mathbf{\Pi_0}||_F^2\}; \\ \mathbf{\Pi_2^n} &= \{\mathbf{\Pi} \in \mathbf{\Pi^n} | ||\hat{\mathbf{\Pi}} - \mathbf{\Pi_0}||_F^2 < ||\tilde{\mathbf{\Pi}} - \mathbf{\Pi_0}||_F^2\}.\end{aligned}$$

Following that, we also divide Eqn. (45) into two sets:

$$\begin{aligned}g(\hat{\mathbf{\Pi}}) - g(\tilde{\mathbf{\Pi}}) &= \sum_{\mathbf{\Pi_0} \in \mathbf{\Pi_1^n}} (||\hat{\mathbf{\Pi}} - \mathbf{\Pi_0}||_F^2 - ||\tilde{\mathbf{\Pi}} - \mathbf{\Pi_0}||_F^2)||\mathbf{\Pi_0 \hat{A}} - \mathbf{\hat{B} \Pi_0}||_F^2 \\ &\quad - \sum_{\mathbf{\Pi_0} \in \mathbf{\Pi_2^n}} (||\tilde{\mathbf{\Pi}} - \mathbf{\Pi_0}||_F^2 - ||\hat{\mathbf{\Pi}} - \mathbf{\Pi_0}||_F^2)||\mathbf{\Pi_0 \hat{A}} - \mathbf{\hat{B} \Pi_0}||_F^2 \\ &\leq ||\tilde{\mathbf{\Pi}} - \hat{\mathbf{\Pi}}||_F^2 \sum_{\mathbf{\Pi_0} \in \mathbf{\Pi_1^n}} ||\mathbf{\Pi_0 \hat{A}} - \mathbf{\hat{B} \Pi_0}||_F^2.\end{aligned} \tag{46}$$

where the last inequality holds due to the triangular inequality $||\hat{\mathbf{\Pi}} - \mathbf{\Pi_0}||_F^2 - ||\tilde{\mathbf{\Pi}} - \mathbf{\Pi_0}||_F^2 \leq ||\tilde{\mathbf{\Pi}} - \hat{\mathbf{\Pi}}||_F^2$ and the term $\sum_{\mathbf{\Pi_0} \in \mathbf{\Pi_2^n}} (||\tilde{\mathbf{\Pi}} - \mathbf{\Pi_0}||_F^2 - ||\hat{\mathbf{\Pi}} - \mathbf{\Pi_0}||_F^2)||\mathbf{\Pi_0}\hat{\mathbf{A}} - \hat{\mathbf{B}}\mathbf{\Pi_0}||_F^2$ is positive. Then we have

$$\frac{g(\hat{\mathbf{\Pi}}) - g(\tilde{\mathbf{\Pi}})}{g(\tilde{\mathbf{\Pi}})} = \frac{(||\tilde{\mathbf{\Pi}} - \hat{\mathbf{\Pi}}||_F^2) \sum_{\mathbf{\Pi_0} \in \mathbf{\Pi_1^n}} ||\mathbf{\Pi_0}\hat{\mathbf{A}} - \hat{\mathbf{B}}\mathbf{\Pi_0}||_F^2}{\sum_{\mathbf{\Pi_0} \in \mathbf{\Pi^n}} ||\tilde{\mathbf{\Pi}} - \mathbf{\Pi_0}||_F^2 ||\mathbf{\Pi_0}\hat{\mathbf{A}} - \hat{\mathbf{B}}\mathbf{\Pi_0}||_F^2} \leq \frac{2\beta n \sum_{\mathbf{\Pi_0} \in \mathbf{\Pi^n}} ||\mathbf{\Pi_0}\tilde{\mathbf{A}} - \tilde{\mathbf{B}}\mathbf{\Pi_0}||_F^2}{\sum_{\mathbf{\Pi_0} \in \mathbf{\Pi^n}} ||\tilde{\mathbf{\Pi}} - \mathbf{\Pi_0}||_F^2 ||\mathbf{\Pi_0}\tilde{\mathbf{A}} - \tilde{\mathbf{B}}\mathbf{\Pi_0}||_F^2}. \tag{47}$$

where $||\tilde{\mathbf{\Pi}} - \hat{\mathbf{\Pi}}||_F^2 = 2\beta n$ and $\beta \in [0, 1]$ is the ratio between the number of mistakenly matched nodes and that of all the nodes. The last inequality in (47) holds because $\mathbf{\Pi_1^n} \subset \mathbf{\Pi^n}$.

Now we divide the sum $\sum_{\mathbf{\Pi_0} \in \mathbf{\Pi^n}} ||\mathbf{\Pi_0}\hat{\mathbf{A}} - \hat{\mathbf{B}}\mathbf{\Pi_0}||_F^2$ into two parts:

$$D_1 = \sum_{k \leq \rho n} \sum_{\mathbf{\Pi_0} \in \mathbf{\Pi^n}} ||\mathbf{\Pi_0}\hat{\mathbf{A}} - \hat{\mathbf{B}}\mathbf{\Pi_0}||_F^2;$$

$$D_2 = \sum_{\rho n < k \leq n} \sum_{\mathbf{\Pi_0} \in \mathbf{\Pi^n}} ||\mathbf{\Pi_0}\hat{\mathbf{A}} - \hat{\mathbf{B}}\mathbf{\Pi_0}||_F^2.$$

where ρ is any real number in $[0, 1]$ and we assume that ρn is an integer.[6]

For D_1, in average case we can obtain

$$D_1 \leq \sum_{i=1}^{\rho n} \sum_{\mathbf{\Pi_0} \in \mathbf{\Pi^n}} ||\mathbf{\Pi_0}\hat{\mathbf{A}} - \hat{\mathbf{B}}\mathbf{\Pi_0}||_F^2 \leq \sum_{i=1}^{\rho n} \prod_{j=1}^{i} 2(n - j + 1) \leq \sum_{i=1}^{\rho n} (2n)^i = 2n \frac{(2n)^{\rho n} - 1}{2n - 1} \approx (2n)^{\rho n}.$$

For D_2, according to Inequality (43), in average case we can get

$$D_2 \geq \sum_{k=\rho n+1}^{n} \prod_{j=1}^{k} (n - j + 1) = \sum_{k=\rho n+1}^{n} \frac{n!}{(n-k)!} \geq \sum_{k=\rho n+1}^{n} \frac{n!}{((1-\rho)n)!} = (1-\rho)n \frac{n!}{((1-\rho)n)!}.$$

Note that if we set $\rho = \Omega(1) = c_0$, where $c_0 \to 1$, then $\rho \to 1$ and

$$D_2 \geq c_0 \frac{n!}{c_0!} = cn! \sim c\sqrt{2\pi n}(\frac{n}{e})^n,$$

where c is a constant and the last step holds due to the Stirling's formula. Therefore we can upper bound $\frac{D_2}{D_1}$ as

$$\frac{D_2}{D_1} \geq c \frac{\sqrt{2\pi n}(\frac{n}{e})^n}{(2n)^{\rho n}} = c\sqrt{2\pi n} \left(\frac{n^{1-\rho}}{2^\rho e} \right)^n.$$

Then if ρ is a constant which approaches 1 but does not equal to 1, then we find that when $n \to \infty$, D_2 is of higher order of n than D_1. Therefore we can easily verify that in the denominator of the last term in Inequality (47), $\sum_{\rho n < k \leq n} \sum_{\mathbf{\Pi_0} \in \mathbf{\Pi^n}} ||\tilde{\mathbf{\Pi}} - \mathbf{\Pi_0}||_F^2 ||\mathbf{\Pi_0}\hat{\mathbf{A}} - \hat{\mathbf{B}}\mathbf{\Pi_0}||_F^2$ is of higher order of n than $\sum_{k \leq \rho n} \sum_{\mathbf{\Pi_0} \in \mathbf{\Pi^n}} ||\tilde{\mathbf{\Pi}} - \mathbf{\Pi_0}||_F^2 ||\mathbf{\Pi_0}\hat{\mathbf{A}} - \hat{\mathbf{B}}\mathbf{\Pi_0}||_F^2$, since for $k_1 > \rho n$ and $k_2 < \rho n$, $\mathbf{\Pi_1'} \in S_{k_1}(\tilde{\mathbf{\Pi}})$ and $\mathbf{\Pi_2'} \in S_{k_2}(\tilde{\mathbf{\Pi}})$, we have $||\mathbf{\Pi_1'} - \tilde{\mathbf{\Pi}}||_F^2 \geq ||\mathbf{\Pi_2'} - \tilde{\mathbf{\Pi}}||_F^2$. Therefore, we can leave the term with highest order of n in the denominator and numerator in the last term in Inequality (47) when $n \to \infty$ and thus we can obtain

$$\frac{2\beta n \sum_{\mathbf{\Pi_0} \in \mathbf{\Pi^n}} ||\mathbf{\Pi_0}\hat{\mathbf{A}} - \hat{\mathbf{B}}\mathbf{\Pi_0}||_F^2}{\sum_{\mathbf{\Pi_0} \in \mathbf{\Pi^n}} ||\tilde{\mathbf{\Pi}} - \mathbf{\Pi_0}||_F^2 ||\mathbf{\Pi_0}\hat{\mathbf{A}} - \hat{\mathbf{B}}\mathbf{\Pi_0}||_F^2} \approx \frac{2\beta n \sum_{\rho n < k \leq n} \sum_{\mathbf{\Pi_0} \in S_k(\tilde{\mathbf{\Pi}})} ||\mathbf{\Pi_0}\hat{\mathbf{A}} - \hat{\mathbf{B}}\mathbf{\Pi_0}||_F^2}{\sum_{\rho n < k \leq n} \sum_{\mathbf{\Pi_0} \in S_k(\tilde{\mathbf{\Pi}})} ||\mathbf{\Pi_0} - \tilde{\mathbf{\Pi}}||_F^2 ||\mathbf{\Pi_0}\hat{\mathbf{A}} - \hat{\mathbf{B}}\mathbf{\Pi_0}||_F^2} \leq \frac{2\beta n}{2\rho n} \frac{\sum_{\rho n < k \leq n} \sum_{\mathbf{\Pi_0} \in S_k(\tilde{\mathbf{\Pi}})} ||\mathbf{\Pi_0}\hat{\mathbf{A}} - \hat{\mathbf{B}}\mathbf{\Pi_0}||_F^2}{\sum_{\rho n < k \leq n} \sum_{\mathbf{\Pi_0} \in S_k(\tilde{\mathbf{\Pi}})} ||\mathbf{\Pi_0}\hat{\mathbf{A}} - \hat{\mathbf{B}}\mathbf{\Pi_0}||_F^2} = \frac{\beta}{\rho}.$$

Thus we have the approximation ratio

$$\frac{g(\tilde{\mathbf{\Pi}})}{g(\hat{\mathbf{\Pi}})} \geq \frac{1}{1 + \frac{\beta}{\rho}} \approx \frac{1}{1 + \beta} \geq \frac{1}{2}.$$

□

Note that in the proof of Theorem 2, we use several times of inequality scaling method to derive the lower bound of approximation ratio, which is 0.5. These inequality scaling may cause this lower bound to be smaller than the real approximation ratio. That is to say, the approximation ratio 0.5 may be even worse than the approximation ratio in the worst case in real situations. For example, in Inequality (47) we directly use

$$\sum_{\mathbf{\Pi_0} \in \mathbf{\Pi_1^n}} ||\mathbf{\Pi_0}\hat{\mathbf{A}} - \hat{\mathbf{B}}\mathbf{\Pi_0}||_F^2 \leq \sum_{\mathbf{\Pi_0} \in \mathbf{\Pi^n}} ||\mathbf{\Pi_0}\hat{\mathbf{A}} - \hat{\mathbf{B}}\mathbf{\Pi_0}||_F^2,$$

which may cause a big gap. Therefore, for a more general situation we have the following corollary.

Corollary 1. *Given the published graph G_1, the auxiliary graph G_2, the parameter set $\boldsymbol{\theta}$ and the weight matrix $\mathbf{W}$, and we let*

$$\chi = \left(\sum_{\mathbf{\Pi_0} \in \mathbf{\Pi_1^n}} ||\mathbf{\Pi_0}\hat{\mathbf{A}} - \hat{\mathbf{B}}\mathbf{\Pi_0}||_F^2 \right) \Big/ \left(\sum_{\mathbf{\Pi_0} \in \mathbf{\Pi^n}} ||\mathbf{\Pi_0}\hat{\mathbf{A}} - \hat{\mathbf{B}}\mathbf{\Pi_0}||_F^2 \right),$$

then in average case, the approximation $g(\tilde{\mathbf{\Pi}})/g(\hat{\mathbf{\Pi}})$ ratio is larger than $\frac{1}{1+\beta\chi}$.

This corollary can be easily proved by slightly changing the form of Eqn. (47). To take an example to illustrate the gap of approximation ratio caused by χ more intuitively, we assume that $\sum_{\mathbf{\Pi} \in \mathbf{\Pi_1^n}} ||\mathbf{\Pi_0}\hat{\mathbf{A}} - \hat{\mathbf{B}}\mathbf{\Pi_0}||_F^2 = \sum_{\mathbf{\Pi} \in \mathbf{\Pi_2^n}} ||\mathbf{\Pi_0}\hat{\mathbf{A}} - \hat{\mathbf{B}}\mathbf{\Pi_0}||_F^2$ [7]. Then $\chi = \frac{1}{2}$ and $(\frac{1}{1+\beta\chi}) > \frac{2}{3}$, which causes the gap of the lower bound of approximation ratio to be $\frac{2}{3} - \frac{1}{2} = \frac{1}{6}$.

Note that we still claim that the approximation ratio is *larger* than $(\frac{1}{1+\beta\chi})$. This is because we eliminate the sum $\sum_{\mathbf{\Pi_0} \in \mathbf{\Pi_2^n}} (||\hat{\mathbf{\Pi}} - \mathbf{\Pi_0}||_F^2 - ||\tilde{\mathbf{\Pi}} - \mathbf{\Pi_0}||_F^2) ||\mathbf{\Pi}\hat{\mathbf{A}} - \hat{\mathbf{B}}\mathbf{\Pi}||_F^2$ in Eqn. (46), which also generates a gap between the lower bound $\frac{1}{1+\beta\chi}$ and the real approximation ratio. We leave it a future direction to find a proper estimation of this gap. However, the current gap still ensures the real approximation ratio strictly larger than $\frac{1}{1+\beta\chi}$, which further strengthens our

[6] If it is not an integer, we can easily modify it by rounding.

[7] This is only a very special situation, which we use it to make an intuitive example to explain how χ causes the gap of approximation ratio. It is not necessarily the same as real situations

claim at the beginning of Section IV-C that the transformation of the original MMSE problem is valid.

APPENDIX D
PROOF OF LEMMA 1

Proof. We know $||\mathbf{\Pi}\hat{\mathbf{A}} - \hat{\mathbf{B}}\mathbf{\Pi}||_F = ||\mathbf{W} \circ (\mathbf{\Pi}\mathbf{A} - \mathbf{B}\mathbf{\Pi})||_F$, thus we only need to prove that

$$\mathbf{W} \circ \mathbf{\Pi}\mathbf{A} = \mathbf{\Pi}(\mathbf{W} \circ \mathbf{A}).$$

Note that the value of w_{ij} depends only on $p_{\boldsymbol{C_i}\boldsymbol{C_j}}$, s_1 and s_2. For some given nodes i, j, s, t, if $\boldsymbol{C_i} = \boldsymbol{C_s}$ and $\boldsymbol{C_j} = \boldsymbol{C_t}$, then $w_{ij} = w_{st}$. This fact tells that the weight for every node pair stays invariant as long as the node pair keeps the same community structure. Therefore, since $\mathbf{\Pi}$ follows that for every i, j, if $\mathbf{\Pi}(i, j) = 1$, then $\boldsymbol{C_i} = \boldsymbol{C_j}$, there would be $(\mathbf{W} \circ \mathbf{\Pi}\mathbf{A})_{ij} = w_{ij} \cdot \sum_{k=1}^{n} \mathbf{\Pi}_{ik}\mathbf{A}_{kj} = \sum_{k=1}^{n} \mathbf{\Pi}_{ik}(w_{kj} \cdot \mathbf{A}_{kj}) = (\mathbf{\Pi}(\mathbf{W} \circ \mathbf{A}))_{ij}$. Hence, $\mathbf{W} \circ \mathbf{\Pi}\mathbf{A} = \mathbf{\Pi}\mathbf{W} \circ \mathbf{A}$. Combining the assumption that $\mathbf{\Pi}\hat{\mathbf{A}} = \mathbf{W} \circ \mathbf{\Pi}\mathbf{A}$, we have $\hat{\mathbf{A}} = \mathbf{W} \circ \mathbf{A}$. Similarly, $\hat{\mathbf{B}} = \mathbf{W} \circ \mathbf{B}$. The lemma holds with these derivations. □

APPENDIX E
PROOF OF LEMMA 2

Proof. First we verify that $F_1(\mathbf{\Pi})$ is a convex function. One of the sufficient and necessary condition for a function whose variable is matrix is convex is that the Hessian matrix of this function is positive semi-definite. The Hessian matrix of $F(\mathbf{\Pi})$ can be obtained by taking the second derivative over $\mathbf{\Pi}$ on $F(\mathbf{\Pi})$, we denote it as $\nabla^2 F(\mathbf{\Pi})$. Therefore we can obtain the Hessian matrix of $F_1(\mathbf{\Pi})$ by

$$\nabla^2 F_1(\mathbf{\Pi}) = \nabla^2 F_0(\mathbf{\Pi}) - \lambda_{min}\mathbf{I}.$$

where $\mathbf{I}$ is the identity matrix.[8] Note that λ_{min} is the minimum eigenvalue of $\nabla^2 F_0(\mathbf{\Pi})$, therefore all the eigenvalues of $\nabla^2 F_0(\mathbf{\Pi}) - \lambda_{min}\mathbf{I}$ are equal to or larger than 0. Hence $\nabla^2 F_1(\mathbf{\Pi})$ is a nonnegative definite matrix and $F_1(\mathbf{\Pi})$ is a convex function.

Meanwhile, one of the sufficient and necessary conditions for a function whose variable is matrix is concave is that the Hessian matrix of this function is negative semi-definite. Similar to the analysis of $F_1(\mathbf{\Pi})$, we can verify that $F_2(\mathbf{\Pi})$ is a concave function. Thus we complete the proof. □

APPENDIX F
PROOF OF LEMMA 3

For the inner loop part,we focus on $F_\xi(\mathbf{\Pi_{k+1}})$ and $F_\xi(\mathbf{\Pi}^\xi)$. Since $\mathbf{\Pi_{k+1}} = \mathbf{\Pi_k} + \gamma_k(\mathbf{X}_\perp - \mathbf{\Pi_k})$, according to Taylor's Theorem,

$$\begin{aligned} F_\xi(\mathbf{\Pi_{k+1}}) &= F_\xi(\mathbf{\Pi_k} + \gamma_k(\mathbf{X}_\perp - \mathbf{\Pi_k})) \\ &= F_\xi(\mathbf{\Pi_k}) + \gamma_k \mathbf{tr}(\nabla F_\xi^T(\mathbf{\Pi_k})(\mathbf{X}^\perp - \mathbf{\Pi_k})) + \gamma_k \mathbf{R_k} \\ &\leq F_\xi(\mathbf{\Pi_k}) + \gamma_k \mathbf{tr}(\nabla F_\xi^T(\mathbf{\Pi_k})(\mathbf{\Pi}^\xi - \mathbf{\Pi_k})) + \gamma_k \mathbf{R_k}, \end{aligned} \tag{48}$$

where $\gamma_k \mathbf{R_k}$ is the remainder of this Taylor series, and this form makes sense since the remainder must contain a multiplicative factor of γ_k. The last inequality holds since $\mathbf{X}^\perp$ is the minimizer of $\mathbf{tr}(\nabla F_\xi^T(\mathbf{\Pi_k})(\mathbf{\Pi}^\xi - \mathbf{\Pi_k}))$.

In terms of $F_\xi(\mathbf{\Pi}^\xi)$, we have

$$\begin{aligned} F_\xi(\mathbf{\Pi}^\xi) &= F_\xi(\mathbf{\Pi_k} + \mathbf{\Pi}^\xi - \mathbf{\Pi_k}) \\ &= F_\xi(\mathbf{\Pi_k}) + \mathbf{tr}(\nabla F_\xi^T(\mathbf{\Pi_k})(\mathbf{\Pi}^\xi - \mathbf{\Pi_k})) + \mathbf{R'_k}, \end{aligned} \tag{49}$$

where $\mathbf{R'_k}$ is the remainder of this Taylor series.

Combining Eqn. (48) and (49), we can obtain

$$F_\xi(\mathbf{\Pi_{k+1}}) \leq F_\xi(\mathbf{\Pi_k}) + \gamma_k(F_\xi(\mathbf{\Pi}^\xi) - F_\xi(\mathbf{\Pi_k})) + \gamma_k(\mathbf{R_k} - \mathbf{R'_k}). \tag{50}$$

Denote $\Delta\mathbf{R_k} = \mathbf{R_k} - \mathbf{R'_k}$ and by simple transformation of Inequality (50), we obtain

$$F_\xi(\mathbf{\Pi_{k+1}}) - F_\xi(\mathbf{\Pi}^\xi) \leq (1-\gamma_k)(F_\xi(\mathbf{\Pi_k}) - F_\xi(\mathbf{\Pi}^\xi)) + \gamma_k \Delta\mathbf{R_k}. \tag{51}$$

Note that Inequality (51) builds up the relationship between $F_\xi(\mathbf{\Pi_{k+1}})$ and $F_\xi(\mathbf{\Pi_k})$, and we obtain

$$\begin{aligned} &F_\xi(\mathbf{\Pi_{k+1}}) - F_\xi(\mathbf{\Pi}^\xi) \\ &\leq \prod_{i=1}^{k}(1-\gamma_i)(F_\xi(\mathbf{\Pi_1}) - F_\xi(\mathbf{\Pi}^\xi)) + \sum_{i=1}^{k}\gamma_i \prod_{j=1}^{k-i}(1-\gamma_j)\Delta\mathbf{R}_i. \end{aligned} \tag{52}$$

For $F_\xi(\mathbf{\Pi_1}) - F_\xi(\mathbf{\Pi}^\xi)$, note that $\mathbf{\Pi_1} = \mathbf{\Pi}^{\xi-\boldsymbol{\Delta}\xi}$, then

$$\begin{aligned} F_\xi(\mathbf{\Pi}^\xi) &= F_0(\mathbf{\Pi}^\xi) + \xi(n - ||\mathbf{\Pi}^\xi||_F^2) \\ &= F_0(\mathbf{\Pi}^\xi) + (\xi - \Delta\xi)(n - ||\mathbf{\Pi}^\xi||_F^2) - \Delta\xi(n - ||\mathbf{\Pi}^\xi||_F^2) \\ &\geq F_0(\mathbf{\Pi}^{\xi-\boldsymbol{\Delta}\xi}) + (\xi - \Delta\xi)(n - ||\mathbf{\Pi}^{\xi-\boldsymbol{\Delta}\xi}||_F^2) \\ &\quad - \Delta\xi(n - ||\mathbf{\Pi}^\xi||_F^2) \\ &= F_0(\mathbf{\Pi}^{\xi-\boldsymbol{\Delta}\xi}) + \xi(n - ||\mathbf{\Pi}^{\xi-\boldsymbol{\Delta}\xi}||_F^2) \\ &\quad + \Delta\xi(||\mathbf{\Pi}^\xi||_F^2 - ||\mathbf{\Pi}^{\xi-\boldsymbol{\Delta}\xi}||_F^2) \\ &= F_\xi(\mathbf{\Pi}^{\xi-\boldsymbol{\Delta}\xi}) + \Delta\xi(||\mathbf{\Pi}^\xi||_F^2 - ||\mathbf{\Pi}^{\xi-\boldsymbol{\Delta}\xi}||_F^2). \end{aligned} \tag{53}$$

Hence

$$F_\xi(\mathbf{\Pi}^{\xi-\boldsymbol{\Delta}\xi}) - F_\xi(\mathbf{\Pi}^\xi) \leq \Delta\xi(||\mathbf{\Pi}^{\xi-\boldsymbol{\Delta}\xi}||_F^2 - ||\mathbf{\Pi}^\xi||_F^2). \tag{54}$$

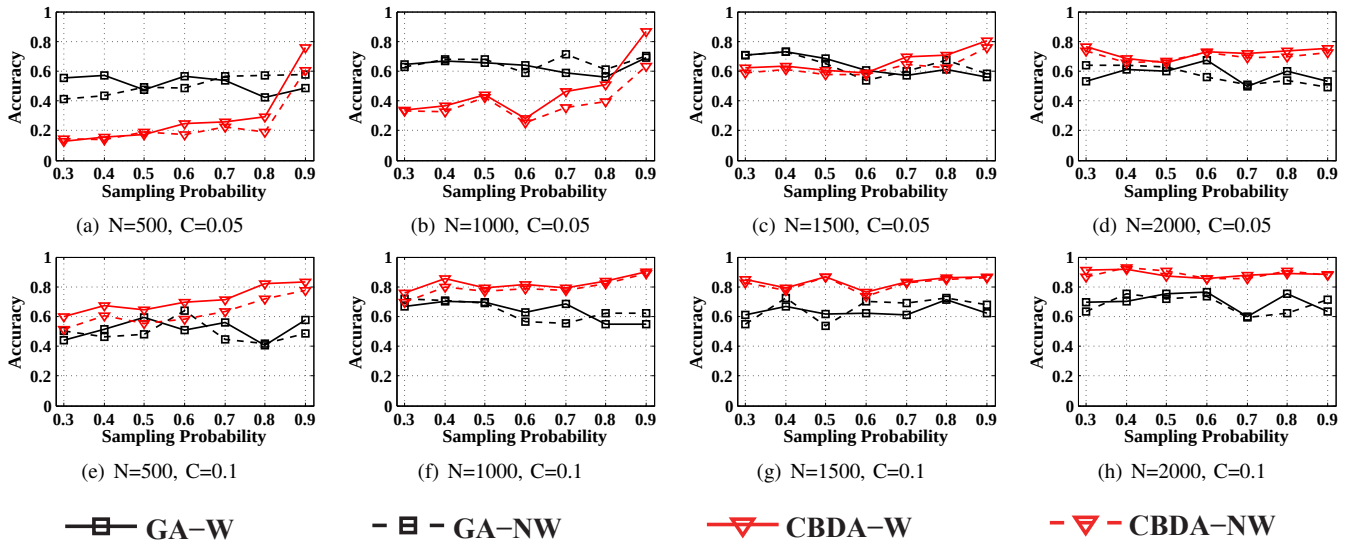
Therefore by combining Inequalities (54) and (52), we can obtain if $\Delta\xi$ is small enough, or if $k \to \infty$, then the term $\prod_{i=1}^{k}(1-\gamma_i)(F_\xi(\mathbf{\Pi_1}) - F_\xi(\mathbf{\Pi}^\xi))$ in last expression of Inequality (52) goes to 0.

For the second term $\sum_{i=1}^{k}\gamma_i \prod_{j=1}^{k-i}(1-\gamma_j)\Delta\mathbf{R}_i$, we note that when $k \to \infty$, then $\forall\epsilon > 0, \exists K > 0, \delta_1 > 0$, when $i > K$, $\gamma_i \prod_{j=1}^{k-i}(1-\gamma_j) < \gamma_i < \frac{\epsilon}{2^{\delta_1 i}}$, and meanwhile when $i \leq K$, $\gamma_k(1-\gamma_j) < \prod_{j=1}^{k-i}(1-\gamma_j) < \frac{\epsilon}{2^{\delta_2 i}}$. Setting $\delta^* = \min\{\delta_1, \delta_2\}$, then we can upper bound the sum $\sum_{i=1}^{k}\gamma_i \prod_{j=1}^{k-i}(1-\gamma_j)\Delta\mathbf{R}_i \leq \sum_{i=1}^{\infty}\frac{\epsilon}{2^{\delta i}} = 0$. Therefore we prove that the inner loop converges.

APPENDIX G
MORE EMPIRICAL RESULTS

We provide Figs. 10, 11, 12 and 13 here to show more detailed empirical results about the effect of weight matrix $\mathbf{W}$ and results on synthetic networks as well as sampled real social networks. These results strengthen our conclusions from the experimental aspect.

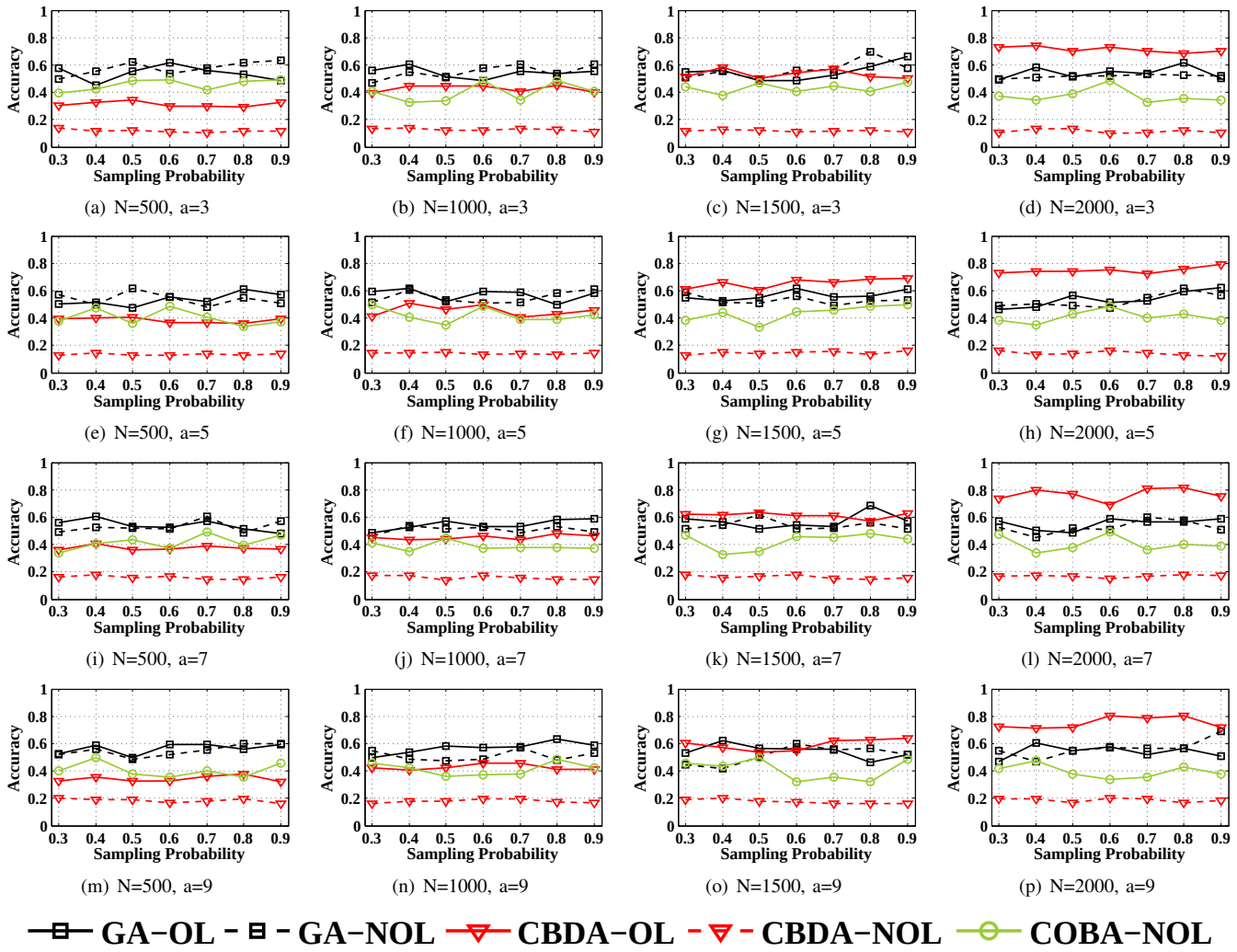
[8]The identity matrix I means all the elements on the diagonal of I are all 1s while others are all 0s. Note that here I is an $n^2 \times n^2$ matrix since the first order derivative of a function whose variable is a matrix is a $n \times n$ matrix, thus the second derivative of F_0 (F_1) is $n^2 \times n^2$ matrix.



(a) N=500, C=0.05 (b) N=1000, C=0.05 (c) N=1500, C=0.05 (d) N=2000, C=0.05

(e) N=500, C=0.1 (f) N=1000, C=0.1 (g) N=1500, C=0.1 (h) N=2000, C=0.1

Fig. 10: Experiments on Weighted and Non-weighted Cost Function.



(a) N=500, a=3 (b) N=1000, a=3 (c) N=1500, a=3 (d) N=2000, a=3

(e) N=500, a=5 (f) N=1000, a=5 (g) N=1500, a=5 (h) N=2000, a=5

(i) N=500, a=7 (j) N=1000, a=7 (k) N=1500, a=7 (l) N=2000, a=7

(m) N=500, a=9 (n) N=1000, a=9 (o) N=1500, a=9 (p) N=2000, a=9

Fig. 11: Experiments on Synthetic Networks with $\eta = 0.05$.

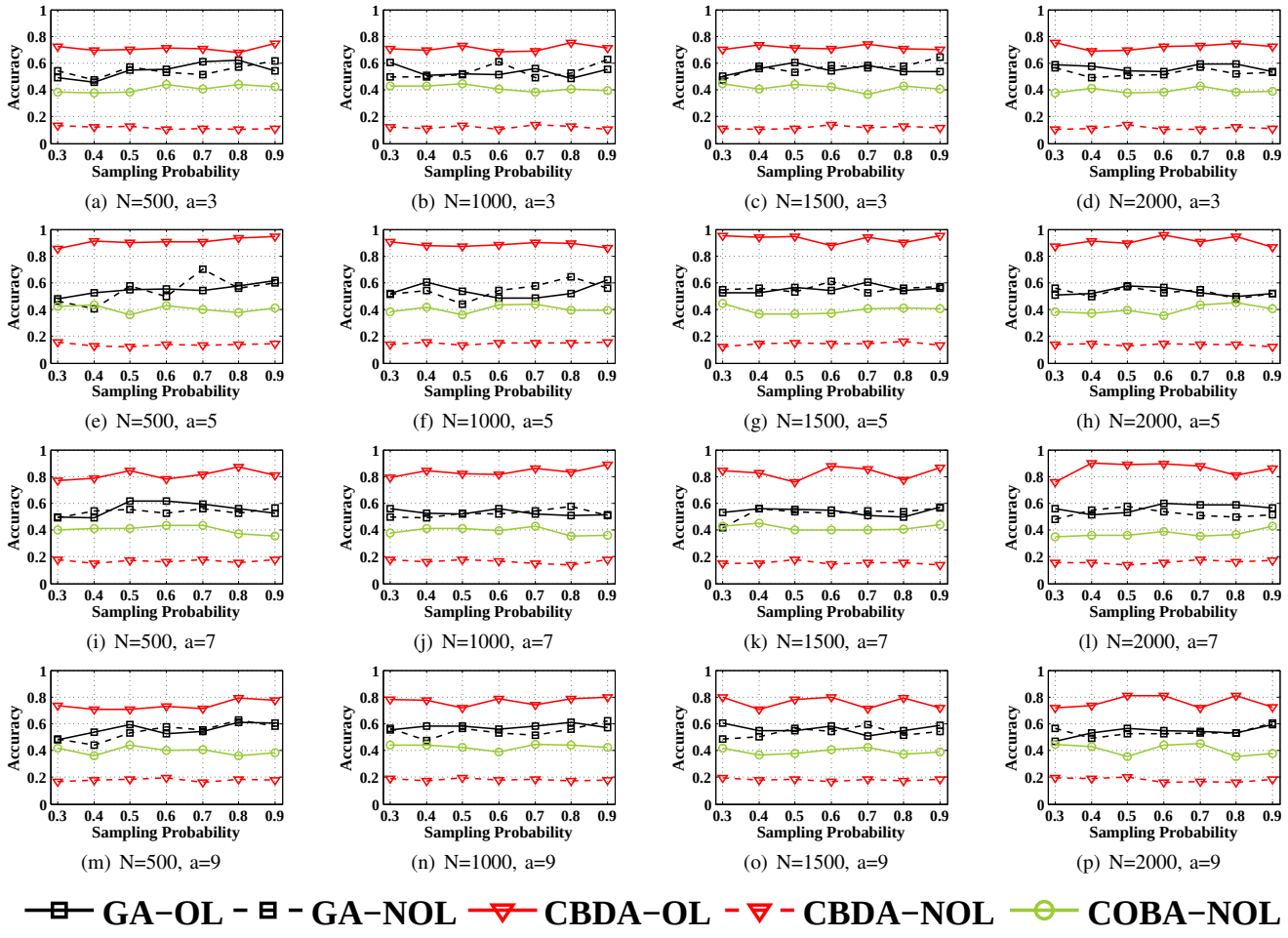


Fig. 12: Experiments on Synthetic Network with $\eta = 0.1$.

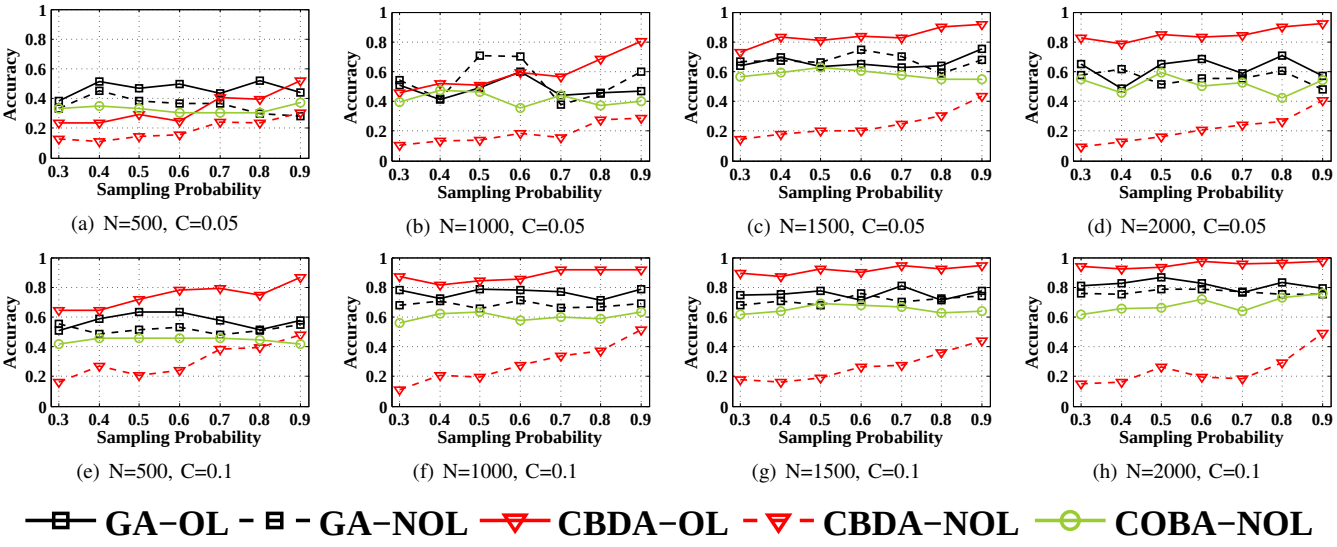


Fig. 13: Experiments on Sampled Real Social Networks.