Percolation Degree in Large Scale Cognitive Radio Networks

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***Abstract*—In large-scale cognitive radio ad hoc networks, the opportunity for a secondary user to communicate with its neighbors is limited by the density of the primary users. We introduce the concept of percolation degree and analyze the performance of secondary network's connectivity with different primary user densities. By applying theories of Poisson point process and continuum percolation, we characterize the relationship between percolation degree of secondary networks and the density of primary users. Furthermore, a tighter upper bound of primary user density is delivered in this literature.**

I. Introduction

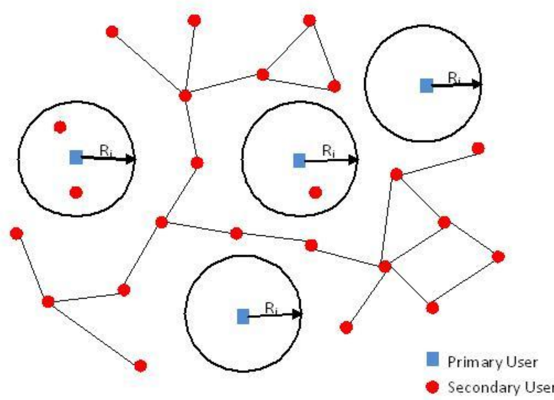

Fig. 1: Topology of Cognitive Radio Network. Primary users have an constant interference radius R_i, secondary users in this region are not allowed to transmit information.

II. System Model

A. Poisson Boolean Model

According to [1], in the Poisson Boolean model, node distribution is a Poisson point process of density λ. We can denote each node as a disk model $B(X_i, R_i)$, where X_i is the node location obeying Poisson distribution and R_i is the disk radius, independent of X_i. First we recall an useful result regarding poisson distribution.

Lemma 1 (Chernoff bound [1]): In a randomly distributed network with node density λ, the probability that there are x nodes falling in a region with area A is

$$Pr(x) = \frac{(\lambda A)^x e^{-\lambda A}}{x!}. \quad (1)$$

B. Network Topology

In cognitive radio networks, secondary network is overlaid with a primary network, their node distributions are uncorrelated while both follow a Poisson Boolean model. We consider this overlaid network in an infinite two dimensional Euclidean space. The network topology is illustrated in Fig.1.

For primary users, they are licensed users behaving regardless of the existence of secondary users, so discussion of primary network characteristics can be ascribed to homogeneous cases. In primary network, nodes are randomly located according to a Poisson point process with density λ_p, given primary node transmission range r_p, primary network can achieve communication connectivity by scattering nodes with density large enough. Throughout this literature, we assume primary network can communicate successfully.

Similarly, secondary nodes are scattered in the network according to a two dimensional Poisson point process with intensity λ_s, which is independent of the primary users. However, differences lies in that secondary users are unlicensed, which means that their communication activities must not interfere licensed users. Secondary users possess the cognitive ability to sense the communication environment, they can adjust their transmission range r_s according to various communication opportunities.

Restricted by the influence of primary network, secondary users have to search for opportunistic spectrum access (OSA) and adjust their communication activities so that the primary user cannot detect the existence of them. In our work, we specify this limitation by defining a rejection radius r_j for each primary users. In rejection regions, secondary users are suppressed to communicate so that primary network communication can be ensured.

C. Continuum Percolation

In two dimensional Poisson point model $B(\lambda, r)$, two nodes are connected if their disks overlap. As two nodes can communicate successfully if they are located within each other's transmission range. Let R_p denote the transmission range for all the nodes, then the node disk radius $R_i = \frac{R_p}{2}$. Define a cluster $C_i(\lambda, r)$ as a group of mutually connected nodes. Let N_i denote the number of the nodes in corresponding cluster, we have the following lemma.

Lemma 2: In a Poisson Boolean model $B(\lambda, r)$, if $\lambda r^2 < p_c$, then

$$Pr(\sup\{N_i\} < \infty) = 1, \tag{2}$$

where p_c is called the critical percolation threshold of two dimensional Poisson Boolean model.

For two models $B(\lambda, r)$ and $B(\lambda_0, r_0)$, if $\lambda r^2 = \lambda_0 r_0^2$, then the associated graphs drawn from the two models are equivalent.

D. Communication Links

As our analysis is specified in cognitive radio networks, we mainly evaluate the connectivity and percolation performance of secondary networks under the restriction of primary users. Every primary user holds an constant rejection radius r_j and secondary network are not allowed to transmit in this region, so there exists a critical primary user density λ_p^* above which secondary connectivity cannot achieve definitely. We specify it in the following theorem.

Theorem 1: In cognitive radio networks employing N_i to represent the node number of secondary users cluster C_i, we define a critical density λ_p^* satisfying $\lambda_p^* r_j^2 = p_c$, then it holds that

$$Pr(\sup\{N_i\} = \infty \ : \ \lambda_p < \lambda_p^*) > 0. \tag{3}$$

Proof: To prove this theorem, firstly we will explain the relationship of that there exists an infinite vacant component and that there is an infinite cluster in the secondary network. As secondary users cannot communicate when located in rejection regions of primary networks, so this infinite cluster of secondary users must be appear in the vacant area outside primary rejection region. On the other hand, as secondary node density in vacant area is $\lambda_s e^{-\lambda_p \pi r_l^2}$, relatively smaller than pure secondary network case, which means available communication users are greatly decreased due to this primary restriction. According to Continuum Percolation, to ensure there is a positive probability that there is an infinite cluster of active secondary nodes, the transmission range of secondary network r_s must satisfy $\lambda_s e^{-\lambda_p \pi r_l^2} r_s^2 > \lambda_c$, which can be easily guaranteed by setting a minimum transmission range of secondary users. With this prerequisite, we can clarify the relationship as following, when there is a vacant component of primary network, there is a positive probability that there exists an infinite cluster of secondary active users.

Then we will prove that when the condition stated in the theorem above is satisfied, there is a vacant component outside the rejection region of all the primary users with probability 1. When $\lambda_p r_j^2 < \lambda_c$, the clusters of rejection disks are in the sub-critical case, which means that the number of all the rejection clusters are finite. Thus for each of such cluster we can always find a vacant area that can encircle the cluster. And a connection of these vacant areas constitutes an infinite vacant component with some finite rejection cluster scattering in it. Recall the explanation in the first part, we complete this proof. ■

As secondary users possess an ability to detect communication opportunities, they can adjust their transmission range according to the distribution density of the primary user. When there are many primary users in the network, few vacant space are left for secondary users to utilize. Every secondary node may only be able to communicate directly with one or two of its neighbors. However when the primary node density decreases, secondary users can detect this opportunity and correspondingly increase their transmission radius to include more neighbors in their transmission disks. We employ the definition of percolation degree to characterize connectivity performance of secondary network.

E. Percolation Degree

Recall percolation theory in [2], when there exists a cluster whose node number tends to infinity, the network is percolated. And such percolated cluster is unique. However, in most cases, the percolated degree, *i.e.* the direct-connected neighbors belonging to the same cluster, of the nodes in this infinite cluster is open to question.

Definition 1 (Percolation Degree): Denote $\mathcal{C}$ as a percolated cluster, for any node $X_i \in \mathcal{C}$, it is connected to several other nodes which are also in this percolated cluster, we call them percolated neighbors and denote the number of them as $D(X_i)$, if the following condition holds

$$Pr(D(X_i) \geq k' \ : \ X_i \in \mathcal{C}) > \alpha.$$

then $k = \max\{k'\}$ is defined as the percolation degree of the network and the network is k-percolated.

III. Main Results

IV. Porosity of Primary Network

As is proved in Theorem 1, when primary node density are larger than λ_p^*, the probability that there exists an infinite cluster in secondary network is zero. In this literature we mainly consider the supercritical phase in secondary network and develop our discussion on condition that $\lambda_p < \lambda_p^*$. As secondary network communications are limited by available spectrum opportunities, the cognitive network characteristics greatly depend on the value of λ_p. When primary users are few, there are sufficient communication opportunities for secondary users, most secondary users are out of the rejection regions of primary nodes and thus communicate without any spectrum constraints. The influence of primary network on secondary network is trivial. However, when primary users increase their occupation in communication channels, secondary users have to succumb to primary dominators, *i.e.* those who are located in the rejection regions have to cease to transmit. Consequently the connectivity of the secondary network are greatly weakened by the increasing number of primary users. First we will explore the porosity of primary network.

Theorem 2: When $\lambda_p < \lambda_p^*$, let $f(L)$ denote the probability density that the distance between any two nodes in the primary network is L, we have

$$f(L) = 2\lambda\pi L e^{-\lambda\pi L^2} \tag{4}$$

Proof: Take X_i as a center, let $N_L(X_i)$ denote the number of neighbors in the disk $B(X_i, L)$, then according to Lemma 1, the probability that there is no nodes located in the disk is

$$Pr(N_L(X_i) = 0) = e^{-\lambda\pi L^2}. \tag{5}$$

Also the probability that there are more than one node located in the annulus between concentric circles with radiuses L and $L + \Delta L$ is

$$Pr(N_{(L+\Delta L)/L}(X_i) \geq 1) = 1 - e^{-\lambda\pi\{(L+\Delta L)^2 - L^2\}} \tag{6}$$

When $\Delta L \to 0$, $Pr(N_{(L+\Delta L)/L}(X_i) \geq 1) \to 2\lambda\pi L\Delta L$. Then we can calculate the probability that the nearest neighbor is of distance L from X_i :

$$\begin{aligned} Pr(L) &= Pr(N_L(X_i) = 0) \cdot Pr(N_{(L+\Delta L)/L}(X_i) \geq 1) \\ &= 2\lambda\pi L\Delta L e^{-\lambda\pi L^2}. \end{aligned} \tag{7}$$

Thus we get the result of Theorem 2. ■

As only those secondary users distributed outside the rejection regions, we call them active nodes, are able to communicate, when the distance between two primary nodes are less than $2r_j$, the overlapped part of their rejection disks are large, thus the secondary network communication can not reach this area. As is demonstrated in Theorem 1, when the primary node density are larger than critical λ_p^*, the secondary network are divided into several isolated clusters with finite size, connectivity of the whole secondary network cannot achieve definitely. When primary node density decreases, average distance between any two primary users will become large, thus more vacant space are available for secondary network to explore spectrum opportunities.

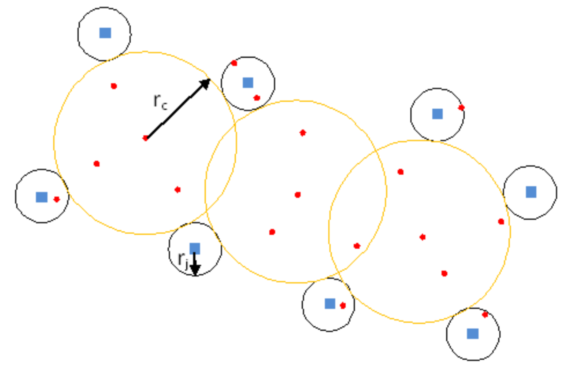

Fig. 2: Communication range of secondary network. When primary node density decreases, the number of active nodes in the neighborhood of an active secondary user increases.

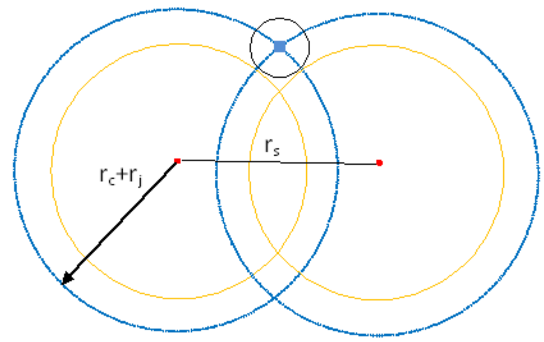

Fig. 3: Variation of communication range in terms of primary network density.

V. Variation of Communication Range for Secondary Network

When the characteristics of primary network distribution change in different scenarios, secondary users are faced with different communication environments. A direct phenomenon is that the number of active nodes in the neighborhood of every secondary node may increase. For an active secondary user, a disk centered at the location of this secondary user in which all the secondary users are active, namely they are all located outside the rejection region of primary network, is called communication region. And the radius of the communication disk is called communication range of this node. In the following part of this section, we will calculate the maximum communication range of secondary users.

VI. Conclusion

The conclusion goes here.

Acknowledgment

References

[1] John Frank Charles Kingman, *Poisson Process*, Oxford University Press, 1993.

[2] R.Meester and R.Roy, *Continuum Percolation*, Cambridge University Press, 1996.