A General Overview on Compressed Sensing

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Abstract—**This paper focus on the general theories of Compressed Sensing in progress and its applications which have shown their potential on imaging and signal processing.**

Index Terms—**Compressed Sensing, Restricted Isometry Property, Magnetic Resonance Imaging (MRI), Compressed Sensing Camera, paper, template.**

I. Introduction

IN this paper we focus on the mathematical theory of Compressed-Sensing (CS)that images with a sparse property can be recovered from randomly undersampled k-space data, provided an appropriate nonlinear recovery scheme is used, and the application of CS, such as MRI and CS camera.

March 29, 2009

II. Theoretical Development in Progress

Shannon's theorem goes that the sampling rate must be at least twice the maximum frequency present in the signal. With the broad success of lossy compression formats, it results in the fact that most of the data previously acquired can be dismissed.

By directly measuring the data information that won't be thrown away, CS asserts far fewer samples or measurements than traditional dogma uses.

A. A General Overview on CS Principle

The acquisition and sampling process can be wrapped up in the following equation.

$$y = \phi x = \phi \varphi s = \Theta s \tag{1}$$

x is the discrete signal and can be viewed as $N \times 1$ column vector in $\mathbb{R}^N$. $\phi$is used for sensing the object x, $\varphi$is used to represent x. s is computed $s = \varphi^T x$. Given x is sparse or compressible, the K largest numbers are located and $N - K$ is dismissed. Θ is an $M \times N$ matrix

So the question is to recover x from $M \approx K$ measurement y. For the successful recovery of sparse x, a sufficient condition restricted isometry property or RIP property is introduced.

$$(1-\delta) \parallel s \parallel_2^2 = \parallel \Theta s \parallel_2^2 = (1+\delta) \parallel s \parallel_2^2 \tag{2}$$

If Θ satisfies, the solution s can be recovered using linear programming l_1.

$$\hat{s} = argmin \parallel s' \parallel_2 such that \Theta s' = y \tag{3}$$

The above is the common view enjoyed to deal with the compressive sampling technique.

The compressive sensing theory is still being challenged around the following several aspects.

B. Sparsity Dictionaries to Use

Sparsity is one of the principle that CS relies on. Strictly defined, a sparse signal can be expanded in a small number of terms. While it can also be defined for signals that can be expanded in a series with significantly decaying coefficients. Given unlabeled set of data, one may find them sparse in a basis found in harmonic analysis.

1) Fourier, Polynomials, etc.
2) all kinds of wavelets and higher dimensional related functions.

Otherwise algorithms are used to find sparse dictionaries data driven dictionaries.

Algorithms for finding the representation of input data learns basis functions that capture higher-level features in the data.

Recent sparse coding development focuses on the computational cost. A class of efficient sparse coding algorithms based on iteratively solving two convex optimization problems is proposed[1]. It's based on alternating optimization over two subsets of the variables. The first subset is an l_1-regularized least squares problem; The second is an $l_2$constrained least squares problem.

Some realization coding can be found Matlab Tool Box from Gabriel Peyre[2]. Sparse coding algorithms Knowledge of specific domain signals enables the ability to build these hopefully small dictionaries.

C. CS Measurement

Incoherence is the other principle that CS relies on. projections on. It's the restriction to the pairs ϕ, the sensing matrix and φ, the representation matrix. To check whether ϕ or the specific measurement matrix allow for sparse solution to be recovered, Restricted Isometry Property or RIP [3]is proposed.

1) Questions around RIP: RIP is a sufficient condition and too strict.

Some papers conform to this property and applies it into application. In "Compressive Coded Aperture Superresolution Image Reconstruction"[4], the author describes the design of coded aperture masks for superresolution image reconstruction from a single, low-resolution, noisy observation image by the generation of coded aperture masks which allow the associated observation matrix to satisfy the RIP.

But there still pops up papers challenging the RIP. In "Sparsest Solutions of Underdetermined Linear Systems via l_q–minimization for $0 \leq q \leq 1$"[5], the author presents a condition on the matrix of an underdetermined linear system which guarantees that the solution of the system with l_q–quasinorm is also the sparsest one, but avoids the Restricted Isometry Constants.

By unifying the different measurement matrices[6], the RIP is broken in RIP(1) and RIP(2).

2) Encoding Matrices: RIP property brings two kinds of encoding. The first follows RIP(1) property and the second follows RIP(2) property.
For RIP(2)

1) Random Fourier Ensemble, that is the measurements are the Fourier coefficient at a randomly selected set Omega of frequencies.
2) Gaussian ensemble, that is the measurements are the coefficients of iid Gaussian variables.
3) Bernoulli ensemble, that is the measurements are the coefficients of iid Bernoulli variables.
4) Toeplitz-structured cs matrices, with entries drawn independently from the same distributions[7]
5) etc.

For RIP(1)

1) Count–Min matrix, the kind of matrices used by the Count–Min algorithm.[8]
2) LDPC encoding matrices, which consist mostly of zeros, where nonzero entries are -1,+1.
3) etc.

The above are all non–deterministic and non-adaptive measurements. Even though non-adaptive is a feature of CS, recent study head towards non–deterministic and adaptive measurement in form o f bayesian approach.

In "Bayesian Compressive Sensing" [8], in addition to estimating the underlying signal x, "error bars" are also estimated. By using knowledge of the error bars, a principled means is provided for determining when a sufficient number of compressive-sensing measurements have been performed, hence, the compressive sensing measurements are optimized adaptively and hence not determined randomly.

The deterministic measurements are also used in encoding matrix. Compared with randomized sub–linear time algorithms which have a small (controllable) probability of failure for each processed signal, deterministic algorithms have it advantage. For example, deterministic sub-linear time sparse Fourier Transform algorithm is developed[9].

In addition, introducing the prior information is also proposed[10], and the author suggests a significant enhancement of CS recoverability.

D. Reconstruction solvers

Early report already showed the possibility of reconstruction[10] from (3): The recovery problem (3) is convex and theoretically justified, to practically solve this program, however it has drawbacks.

1) the iterative interior point methods, although tractable and stable, require a $M \times M$ system of equations to be solved at each step.
2) Because the l_1 norm functional does not distinguish between coefficients at different scales, energy has been shifted from coarse to fine scale.

A slightly different recovery procedure is proposed, which require a small amount of a priori information about signal, but brings desirable results.(3)was modified.
With the development of the CS theory, the reconstruction develops along with the measurement encodings. Nowadays, the booming reconstruction codes include a wide series of techniques: Matching Pursuit/Greedy, Basis Pursuit/Linear Programming, Bayesian, Iterative Thresholding, Proximal.

These many algorithms feature themselves in various aspects. [6]summarizes and group all this algorithmic approaches to sparse signal recovery: geometric and combinatorial and have a sum-up on them.

III. Application for Compressed Sensing

A. Compressed Sensing MRI

Magnetic Resonance Imaging (MRI) is primarily a medical imaging technique most commonly used in radiology to visualize the internal structure and function of the body. Applying CS to MRI offers potentially significant reductions of the scanning time and the patient's exposure to electromagnetic radiation, with benefits for patients and health care economics.

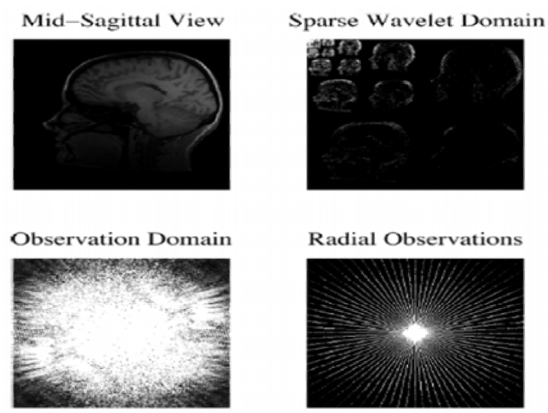

Fig. 1. Compressed sensing applied to medical-resonance imaging (MRI)[11]

In Figure1, the mid-sagittal view(top left) is sparsely represented in the wavelet domain(top right). The MRI scanner takes samples in the 2D Fourier domain(bottom left). Compressed sensing only measures a subset of this data, such as a small number of radial slices(bottom right). From the figure, one can find out that reconstruction algorithms search for the sparsest representation in the wavelet domain consistent with the observation and inverse wavelet transformation enables image reconstruction[12].

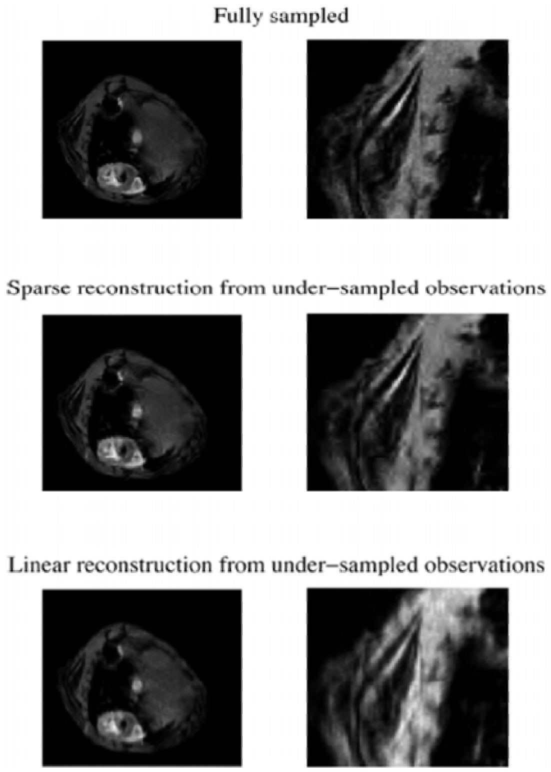

Fig. 2. A frame from an MRI movie of a beating mouse heart[11]

Figure 2 clearly shows the advantages of the compressed sensing approach. The figure top left is a frame from a MRI movie of a beating mouse heart, synchronized to the heart beat, with magnified detail to the right. By using the approach based on only one-fifth of the total number of measurement, we get the reconstruction in the middle panels. At the same time, in the bottom panels, the standard linear-reconstruction algorithms show clear aliasing artifacts and blurring for comparison.

Based on the sparse data representation, people developed the gradient-pursuit algorithm to calculate reconstructions efficiently, especially for large data sets. [13]Noticing that sparsity is a powerful constraint, the fact that the wavelet representation exhibits additional structure shows clearly in Figure 1. More general class of signal models and new algorithms that can optimally exploit the tree structure are being developed. In this way, we might be able to further reduce the number of measurement and improve reconstruction accuracy.[14]

B. Compressed sensing cameras

Allowing a safety margin, if we only need fewer components, such as 100,000, to recover the image, for example, one can simply record the dimensions and location of the square, and note the average colour of that square, and then subtract that average off from the image, leaving a small residual error, there is no need to take a 2 million measurements. CS enables people to measure the largest version of the image. This completely bypasses the need for power-hungry operations such as pixel accumulation and data compression algorithms. In short,the camera converts the incident 2D light field directly into a compressed and encoded 1D bit stream that can be stored of transmitted with no further processing.[15]

The camera should select accurately the hundred thousand of the total wavelet coefficients that have all the interesting information in the image and are able to recover the image. To reach this goal, we construct a linear algebra analogue of a hash function. So we are facing two main problems, that is the issue of noise and the efficient ways to recover the image using the hundred thousand measurements obtained.

There are feasible ways, such as matching pursuit and basis pursuit, to recover the data. And the matching pursuit algorithm tends to be somewhat faster, while the basis pursuit algorithm seems to be more robust with respect to noise.[16]

The single-pixel camera can be proof-of-concept prototype which need us to further develop considering compressed sensing is still a fairly new field.

C. Other Applications

When the original signal is lost or corrupted, Compressed Sensing plays a key role in reconstructing signal even from incomplete and noisy data. For example,if weather, lack of telescope time, or simply the rotation of the earth prevents a complete time-series of data, people need to rely on compressed sensing techniques to measure these phenomena in the time domain and then reconstruct the original signal[17][18].

IV. Conclusion

Regarding the rigorous mathematical results, compressed sensing theory can still wait to be challenged. Lots of optimum algorithms are still being developed. While many applications are only theoretical. But as an abstractive and vigorous sampling method, no one can ignore its potential to impact on so many types of imaging, measurement and signal processing.

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