Optimal determination of source-destination connectivity in random graphs

Roger Fu

June 3rd, 2016

Outline

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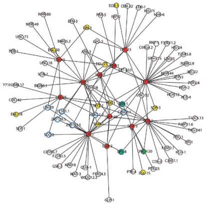
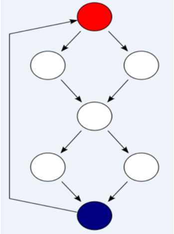
Graphs Are Everywhere

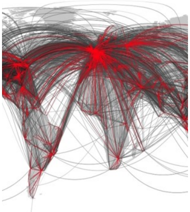


Biological Network

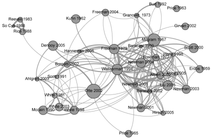


Program Flow

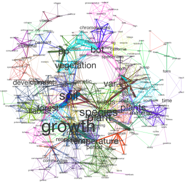
Social Network



Transportation Network



Citation Network



Topic Network

Uncertainty Is Prevalent

"... the real world is always certain; it is our knowledge of it that is sometimes uncertain." [1]

[1] Amihai Motro [Management of Uncertainty in Database Systems]

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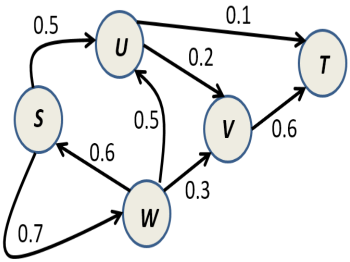


Uncertain Graph
(Edge Uncertainty)

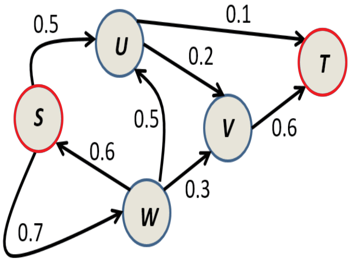
- Communication Networks
- Citation Networks
- Topic Networks

[1] Amihai Motro [Management of Uncertainty in Database Systems]

Outline

Determination of Genuine Relation

- Fundamental problem of determining s-t connectivity.



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Model

Definition (Problem Formulation)

Given a random graph $\mathcal{G}(V, E)$, a probability vector $p = (p_1, p_2, ..., p_{|E|})$, a cost vector $c = (c_1, c_2, ..., c_{|E|})$ and two nodes $s, t \in V$, find an adaptive testing strategy to determine the $s - t$ connectivity while incurring minimum expected cost.

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- The true underlying graph has a product distribution and the expectation of cost is taken over all possible realizations of $\mathcal{G}$.

Adaptive Testing Strategy

- Define the set of states of $\mathcal{G}$ as $S = \{0, 1, *\}^{|E|}$.
- Formally, an adaptive testing strategy is a mapping $S \mapsto E$.
- A strategy terminates by verifying the existence of an $s - t$ path or an $s - t$ cut.

Outline

NP Hardness of the Decision Version

Given a random graph $\mathcal{G}(V, E)$, a probability vector $p = (p_1, p_2, ..., p_{|E|})$, a cost vector $c = (c_1, c_2, ..., c_{|E|})$ and two nodes $s, t \in V$, is there an adaptive testing strategy with expected cost less than k.

A Stronger Result

Theorem

Computing the expected cost of the optimal strategy is #P-hard [2].

Proof

- By reduction from network reliability problem.
- Inspired by Papadimitriou, Christos H., and Mihalis Yannakakis. "Shortest paths without a map."Theoretical Computer Science 84.1 (1991): 127-150.

[2]Valiant, Leslie G. "The complexity of enumeration and reliability problems."

More Words on Class #P

Theorem

Computing the expected cost of the optimal strategy is #P-hard [3].

Leslie Valiant

- Complexity of Counting
- PAC Leaning Model
- Professor at Harvard University
- ACM Turing Award Winner

[3] Valiant, Leslie G. "The complexity of enumeration and reliability problems."

An Even Stronger Result

Theorem

Deciding the next edge to test is NP-hard.

- *By reduction from set cover.*

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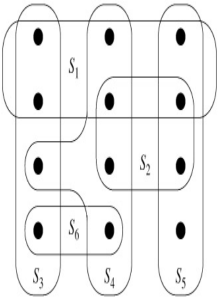

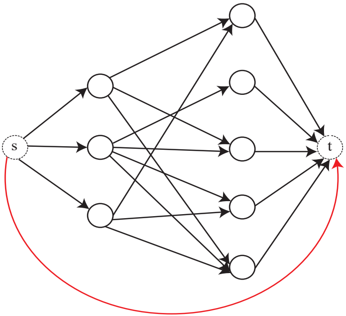

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Markov Decision Process Framework

Definition (Markov Decision Process)

A mathematical model for modeling decision making under uncertain situations. A Markov Decision Process (MDP) model contains:

- A set of possible world states S
- A set of possible actions A
- A reward function $R(s, a)$
- A description T of each action's effect in each state

Markov Property: the effects of an action taken in a state depend only on that state and not one the prior history.

Exact Algorithm Based on Dynamic Programming

Input: Random graph $\mathcal{G}(V, E)$, probability vector $\mathbf{p}$, node s and t
Output: The optimal testing strategy $\mathcal{D}$, that is, a mapping from each partial graph to the next edge to test.
Precompute the elements in the corresponding Finite Horizon Markov Decision Process.
Initialize: $u(s) = 0$, for all $s \in S_N$
for $i = N$ *to* 0 **do**
  for *All* s *in* S_i **do**
    $t_{e^*} = \arg\min_{t_e \in A_s}\{c_e + p_e \times u(S \cdot e) + (1 - p_e) \times u(S \backslash e)\}$
    $u(s) = c_{e^*} + p_{e^*} \times u(S \cdot e^*) + (1 - p_{e^*}) \times u(S \backslash e^*)$
    $\mathcal{D}(s) = t_{e^*}$
  end
end

Exact Algorithm with Exponential Running Time

Outline

A Simple Greedy Approach

Theorem

A strategy that tests the edges following the ascending order of costs is an $O(|E|)$ approximation.

Adaptive Submodular Framework

- An approximation algorithm based on adaptive submodular framework.
- $O(\ln(Q_p Q_c))$ approximation, where Q_p and Q_c are the number of $s-t$ paths and $s-t$ cuts in the graph.
- Q_p and Q_c are polynomial to $|E|$ for most graphs, that is, the algorithm yields a logarithmic approximation in most cases.

A More Sophisticated Approximation Algorithm

Input: Random graph $\mathcal{G}(V, E)$, probability vector $\mathbf{p}$, cost vector $\mathbf{c}$, node s and t
Output: An approximate sequential testing strategy
Initialize: Partial realization ψ, Set of tested edges E_π as empty sets.
Repeat until $g(\psi) = Q_p Q_c$
$e = \arg\max_{e \in E \setminus E_\pi} \{\Delta(e|\psi)\}$.
Set E_π as $E_\pi \cup \{e\}$, test e and observe the outcome.
if $e = 1$ **then**
| $\psi = \psi \cup (e, 1)$
else
| $\psi = \psi \cup (e, 0)$
end

Approximation Algorithm under Adaptive Submodularity Framework

- Fu, Luoyi, Xinbing Wang, and P. R. Kumar. "Optimal determination of source-destination connectivity in random graphs."Mobihoc 2014.
- Arijit Khan and Lei Chen "On Uncertain Graphs Modeling and Queries"
- Valiant, Leslie G. "The complexity of enumeration and reliability problems."SIAM Journal on Computing 8.3 (1979): 410-421.
- Bob Givan and Ron Parr "An introduction to Markov Decision Process"
- Golovin, Daniel, and Andreas Krause. "Adaptive submodularity: Theory and applications in active learning and stochastic optimization."Journal of Artificial Intelligence Research (2011): 427-486.

Thank You!