Lab07-Undecidability

CS363-Computability Theory, Xiaofeng Gao, Spring 2016

$*$ Please upload your assignment to FTP or submit a paper version on the next class

$*$ If there is any problem, please contact: nongeek.zv@gmail.com

$*$ Name:________ StudentId: ________ Email: ________

1. Show that '$W_x = W_y$' is undecidable. (*Hint.* Reduce 'ϕ_x is total' to this problem).

2. Show that 'n is a Fermat number' predicates is partially decidable. (n is a *Fermat number* if $\exists x,\ y,\ z > 0$ such that $x^n + y^n = z^n$)

3. Suppose that $M(x)$ is a predicate and k a total computable function such that $x \in W_x$ iff $M(k(x))$ does not hold. Prove that $M(x)$ is not partially decidable.

4. Proof of Rice's Theorem

   Rice's Theorem says: "Suppose $\varnothing \subsetneq \mathscr{B} \subsetneq \mathscr{C}_1$, then the problem '$\phi_x \in \mathscr{B}$' is undecidable." In the textbook the author used a function construction to show that "$\phi_x \in \mathscr{B}$" is at least as difficult as "$x \in W_x$". After learned the technique of "reduction" to prove the undecidability of given predicates in computability class, students thought of new ways to prove the correctness of Rice's Theorem. The following are two examples:

   (a) One student used the reduction from an undecidable predicate "$\phi_x = \phi_y$". Since $\mathscr{B} \neq \varnothing$, let $\mathscr{B} = \{\phi_y\}$, then "$\phi_x = \phi_y$"$\Leftrightarrow$"$\phi_x \in \mathscr{B}$", which means deciding whether "$\phi_x \in \mathscr{B}$" is at least as hard as deciding whether "$\phi_x = \phi_y$" even with the simplest special case of $\mathscr{B} = \{\phi_y\}$. Since "$\phi_x = \phi_y$" is undecidable, Rice's Theorem is correct correspondingly.

   (b) Another student used the same reduction "$x \in W_x$" as in our textbook. However, he defined a new function $f(x, y) \simeq \begin{cases} g(y) & \text{if } x \notin W_x, \\ undefined & \text{if } x \in W_x. \end{cases}$, where $g \in \mathscr{C}_1 \backslash \mathscr{B}$. The s-m-n theorem provides a total computable function $\kappa(x)$ such that $f(x, y) \simeq \phi_{\kappa(x)}(y)$. Thus we see that $\begin{cases} x \notin W_x & \Rightarrow & \phi_{\kappa(x)} = g, & \text{i.e. } \phi_{\kappa(x)} \notin \mathscr{B} \\ x \in W_x & \Rightarrow & \phi_{\kappa(x)} = f_\varnothing, & \text{i.e. } \phi_{\kappa(x)} \in \mathscr{B} \end{cases}$. So he reduced the problem "$x \in W_x$" to the problem "$\phi_x \in \mathscr{B}$" using the computable function κ.

   Are their proofs correct? If yes, explain your reason. If no, highlight where they made the mistakes. Next, try to design a reduction from some known undecidable predicate other than "$x \in W_x$" to prove Rice's Theorem.