Unlimited Register Machine*

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CS363-Computability Theory

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Outline

What is Effective Procedure

- Methods for addition, multiplication $\cdots$
  - ▷ Given n, finding the nth prime number.
  - ▷ Differentiating a polynomial.
  - ▷ Finding the highest common factor of two numbers $HCF(x, y) \to$ Euclidean algorithm
  - ▷ Given two numbers x, y, deciding whether x is a multiple of y.
- Their implementation requires no ingenuity, intelligence, inventiveness.

Intuitive Definition

An *algorithm* or *effective procedure* is a mechanical rule, or automatic method, or programme for performing some mathematical operations.

Blackbox: input $\longrightarrow$ ■ $\longrightarrow$ output

What is "effective procedure"?

An Example: Consider the function $g(n)$ defined as follows:

$$g(n) = \begin{cases} 1, & \text{if there is a run of exactly } n \text{ consecutive } 7's \\ & \text{in the decimal expansion of } \pi, \\ 0, & \text{otherwise.} \end{cases}$$

Question: Is $g(n)$ effective?

▷ The answer is unknown $\neq$ the answer is negative.

Other Examples:

- *Theorem Proving* is in general not effective/algorithmic.
- *Proof Verification* is effective/algorithmic.

Algorithm

An algorithm is a procedure that consists of a finite set of *instructions* which, given an *input* from some set of possible inputs, enables us to obtain an *output* through a systematic execution of the instructions that *terminates* in a finite number of steps.

Computable Function

When an algorithm or effective procedure is used to calculate the value of a numerical function then the function in question is effectively calculable (or algorithmically computable, effectively computable, computable).

Examples:

- $HCF(x,y)$ is computable;
- $g(n)$ is non-computable.

Unlimited Register Machine

- An Unlimited Register Machine (URM) is an idealized computer.
  - ▷ No limitation in the size of the numbers it can receive as input.
  - ▷ No limitation in the amount of working space available.
  - ▷ Inputs and outpus are restricted to natural numbers. (coding for others)
- From Shepherdson & Sturgis [1963]'s description.
  - ▷ Shepherdson, J. C. & Sturgis, H.E., Computability of Recursive Functions, *Journal of Association for Computing Machinery (Journal of ACM)*, 10, 217-55, 1963.

Register

A URM has an infinite number of register labeled $R_1, R_2, R_3, \ldots$.

R_1	R_2	R_3	R_4	R_5	R_6	R_7	$\cdots$
r_1	r_2	r_3	r_4	r_5	r_6	r_7	$\cdots$

Every register can hold a *natural number* at any moment.

The registers can be equivalently written as for example

$$[r_1 r_2 r_3]_1^3 [r_4]_4^4 [r_5 r_6 r_7 \ldots]_5^\infty$$

or simply

$$[r_1, r_2, r_3]_1^3 [r_4]_4^4 [r_5, r_6, r_7]_5^7.$$

Program

A URM also has a program, which is a finite list of instructions.

An instruction is a recognized simple operations (calculation with numbers) to alter the contents of the registers. $(I_1, \cdots, I_s)$

Instruction

Type	Instruction	Response of the URM
Zero	$Z(n)$	Replace r_n by 0. ($0 \rightarrow R_n$, or $r_n := 0$)
Successor	$S(n)$	Add 1 to r_n. ($r_n + 1 \rightarrow R_n$, or $r_n := r_n + 1$)
Transfer	$T(m, n)$	Copy r_m to R_n. ($r_m \rightarrow R_n$, or $r_n := r_m$)
Jump	$J(m, n, q)$	If $r_m = r_n$, go to the q-th instruction; otherwise go to the next instruction.

$Z(n)$, $S(n)$, $T(m, n)$ are arithmetic instructions.

Configuration and Instructions

Example: The initial registers are:

R_1	R_2	R_3	R_4	R_5	R_6	R_7	$\cdots$
9	7	0	0	0	0	0	$\ldots$

The program is:

$I_1 : J(1, 2, 6)$
$I_2 : S(2)$
$I_3 : S(3)$
$I_4 : J(1, 2, 6)$
$I_5 : J(1, 1, 2)$
$I_6 : T(3, 1)$

Configuration and Computation

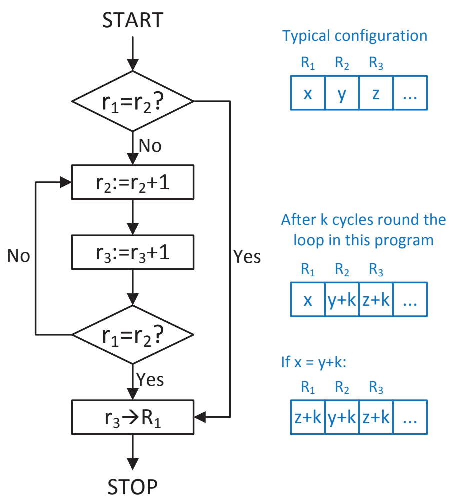
Configuration:
the contents of the registers + the current instruction number.

Initial configuration, computation, final configuration.

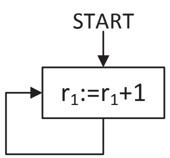
Operation of URM under a program P

- $P = \{I_1, I_2, \cdots, I_s\} \to$ URM
- URM starts by obeying instruction I_1
- When URM finishes obeying I_k, it proceeds to the next instruction in the computation,
  - if I_k is not a jump instruction, then the next instruction is I_{k+1};
  - if $I_k = J(m, n, q)$ then next instruction is (1) I_q, if $r_m = r_n$; or (2) I_{k+1}, otherwise.
- Computation stops when the next instruction is I_v, where $v > s$.
  - if $k = s$, and I_s is an arithmetic instruction;
  - if $I_k = J(m, n, q)$, $r_m = r_n$ and $q > s$;
  - if $I_k = J(m, n, q)$, $r_m \neq r_n$ and $k = s$.

Flow Diagram



- $J(m, m, q)$ is alertunconditional jump
- Computations that never stop



Some Notation

Suppose P is the program of a URM and $a_1, a_2, a_3, \ldots$ are the numbers stored in the registers.

- $P(a_1, a_2, a_3, \ldots)$ is the initial configuration.
- $P(a_1, a_2, a_3, \ldots)\downarrow$ means that the computation converges.
- $P(a_1, a_2, a_3, \ldots)\uparrow$ means that the computation diverges.
- $P(a_1, a_2, \ldots, a_m)$ is $P(a_1, a_2, \ldots, a_m, 0, 0, \ldots)$.

URM-Computable Function

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What does it mean that a URM computes a (partial) n-ary function f?

Let P be the program of a URM and $a_1, \ldots, a_n, b \in \mathbb{N}$. When computation $P(a_1, \ldots, a_n)$ converges to b if $P(a_1, \ldots, a_n) \downarrow$ and $r_1 = b$ in the final configuration. We write $P(a_1, \ldots, a_n) \downarrow b$.

- P URM-computes f if, for all $a_1, \ldots, a_n, b \in \mathbb{N}$,

$$P(a_1, \ldots, a_n) \downarrow b \text{ iff } f(a_1, \ldots, a_n) = b$$

- Function f is URM-computable if there is a program that URM-computes f.
- (We abbreviate "URM-computable" to "computable")

Computable Functions

Let

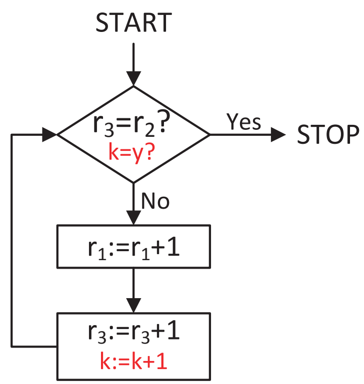
$\mathscr{C}$ be the set of computable functions and

$\mathscr{C}_n$ be the set of n-ary computable functions.

Examples

Construct a URM that computes $x + y$.

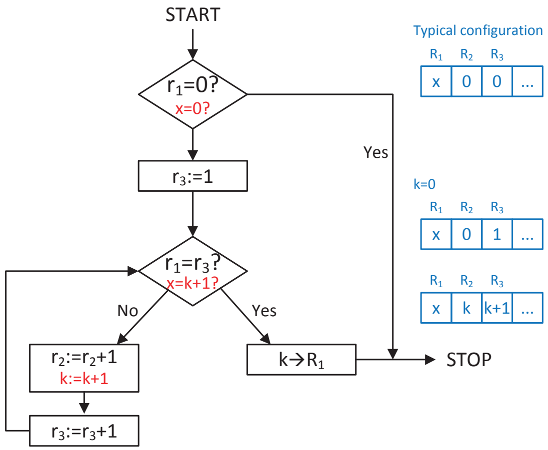
$I_1 : J(3, 2, 5)$
$I_2 : S(1)$
$I_3 : S(3)$
$I_4 : J(1, 1, 1)$



Examples

Construct a URM that computes $x \dot{-} 1 = \begin{cases} x-1, & \text{if } x > 0, \\ 0, & \text{if } x = 0. \end{cases}$

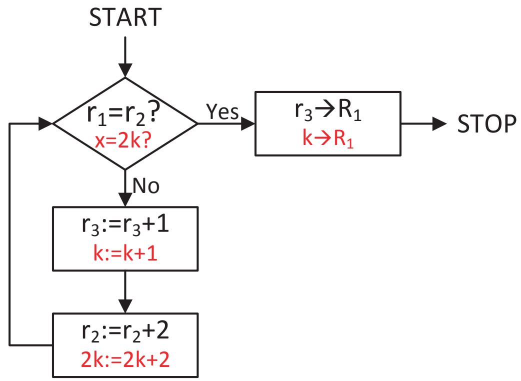
$I_1 : J(1,4,8)$
$I_2 : S(3)$
$I_3 : J(1,3,7)$
$I_4 : S(2)$
$I_5 : S(3)$
$I_6 : J(1,1,3)$
$I_7 : T(2,1)$



Examples

Construct a URM that computes $x \div 2 = \begin{cases} x/2, & \text{if } x \text{ is even}, \\ \text{undefined}, & \text{if } x \text{ is odd}. \end{cases}$

$I_1 : J(1,2,6)$
$I_2 : S(3)$
$I_3 : S(2)$
$I_4 : S(2)$
$I_5 : J(1,1,1)$
$I_6 : T(3,1)$



Function Defined by Program

Given any program P and $n \geq 1$, by thinking of the effect of P on initial configurations of the form $a_1, \cdots, a_n, 0, 0, \cdots$, there is a unique n-ary function that P computes, denoted by $f_P^{(n)}$.

$$f_P^{(n)}(a_1, \ldots, a_n) = \begin{cases} b, & \text{if } P(a_1, \ldots, a_n) \downarrow b, \\ \text{undefined}, & \text{if } P(a_1, \ldots, a_n) \uparrow. \end{cases}$$

Predicate and Decision Problem

The value of a predicate is either ‘true’ or ‘false’.

The answer of a *decision problem* is either ‘yes’ or ‘no’.

Example: Given two numbers x, y, check whether x is a multiple of y.
Input: x, y;
Output: ‘Yes’ or ‘No’.

The operation amounts to calculation of the function

$$f(x,y) = \begin{cases} 1, & \text{if } x \text{ is a multiple of } y, \\ 0, & \text{if otherwise.} \end{cases}$$

Thus the property or predicate ‘x is a multiple of y’ is algorithmically or effectively decidable, or just decidable if function f is computable.

Decidable Predicate and Decidable Problem

Suppose that $M(x_1, \ldots, x_n)$ is an n-ary predicate of natural numbers. The characteristic function $c_M(\mathbf{x})$, where $\mathbf{x} = x_1, \ldots, x_n$, is given by

$$f_P^{(n)}(a_1, \ldots, a_n) = \begin{cases} 1, & \text{if } M(\mathbf{x}) \text{ holds,} \\ 0, & \text{if otherwise.} \end{cases}$$

The predicate $M(\mathbf{x})$ is decidable if c_M is computable; it is undecidable otherwise.

Computability on other Domains

Suppose D is an object domain. A coding of D is an explicit and effective injection $\alpha : D \to \mathbb{N}$. We say that an object $d \in D$ is coded by the natural number $\alpha(d)$.

A function $f : D \to D$ extends to a numeric function $f^* : \mathbb{N} \to \mathbb{N}$. We say that f is computable if f^* is computable.

$$f^* = \alpha \circ f \circ \alpha^{-1}$$

Example

Consider the domain $\mathbb{Z}$. An explicit coding is given by the function α where

$$\alpha(n) = \begin{cases} 2n, & \text{if } n \geq 0, \\ -2n-1, & \text{if } n < 0. \end{cases}$$

Then α^{-1} is given by

$$\alpha^{-1}(m) = \begin{cases} \frac{1}{2}m, & \text{if } m \text{ is even,} \\ -\frac{1}{2}(m+1), & \text{if } m \text{ is odd.} \end{cases}$$

Example (Continued)

Consider the function $f(x) = x - 1$ on $\mathbb{Z}$, then $f^* : \mathbb{N} \to \mathbb{N}$ is given by

$$f^*(x) = \begin{cases} 1 & \text{if } x = 0 \text{ (i.e. } x = \alpha(0)), \\ x-2 & \text{if } x > 0 \text{ and } x \text{ is even (i.e. } x = \alpha(n), n > 0), \\ x+2 & \text{if } x \text{ is odd (i.e. } x = \alpha(n), n < 0). \end{cases}$$

It is a routine exercise to write a program that computes f^*, hence $x - 1$ is a computable function on $\mathbb{Z}$.

Remark

Register Machines are more advanced than Turing Machines.

Register Machine Models can be classified into three groups:

- CM (Counter Machine Model).
- RAM (Random Access Machine Model).
- RASP (Random Access Stored Program Machine Model).

The Unlimited Register Machine Model belongs to the CM class.

Finiteness

Every URM uses only a fixed finite number of registers, no matter how large an input number is.

This is a fine property of Counter Machine Model.

Sequential Composition

Given Programs P and Q, how do we construct the sequential composition $P; Q$?

The jump instructions of P and Q must be modified.

Standard Form: A program $P = I_1, \ldots, I_s$ is in *standard form* if, for every jump instruction $J(m, n, q)$ we have $q \leq s + 1$.

Lemma

For any program P there is a program P^* in standard form such that any computation under P^* is identical to the corresponding computation under P. In particular, for any $a_1, \cdots, a_n$, b,

$$P(a_1, \cdots, a_n) \downarrow b \text{ if and only if } P^*(a_1, \cdots, a_n) \downarrow b,$$

and hence $f_P^{(n)} = f_{P^*}^{(n)}$ for every $n > 0$.

Proof

Suppose that $P = I_1, I_2, \cdots, I_s$. Put $P^* = I_1^*, I_2^*, \cdots, I_s^*$ where

if I_k is not a jump instruction, then $I_k^* = I_k$;

if I_k is not a jump instruction, then $I_k^* = \begin{cases} I_k & \text{if } q \leq s+1, \\ J(m,n,s+1) & \text{if } q > s+1. \end{cases}$

Join/Concatenation

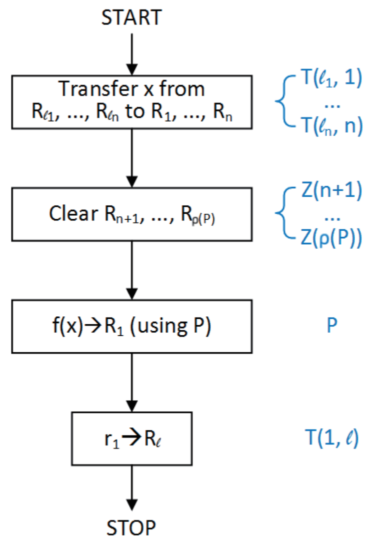
Let P and Q be programs of lengths s, t respectively, in standard form. The *join* or *concatenation* of P and Q, written PQ or ${}^{P}_{Q}$, is a program $I_1, I_2, \cdots, I_s, I_{s+1}, \cdots, I_{s+t}$ where $P = I_1, \cdots, I_s$ and the instructions $I_{s+1}, \cdots, I_{s+t}$ are the instructions of Q with each jump $J(m,n,q)$ replaced by $J(m,n,s+q)$.

Program as Subroutine

Suppose the program P computes f.

Let $\rho(P)$ be the least number i such that the register R_i is not used by the program P.

The notation $P[l_1, \ldots, l_n \to l]$ stands for the following program:



$$\begin{array}{lcl} I_1 & : & T(l_1, 1) \\ & \vdots & \\ I_n & : & T(l_n, n) \\ I_{n+1} & : & Z(n+1) \\ & \vdots & \\ I_{\rho(P)} & : & Z(\rho(P)) \\ \_ & : & P \\ \_ & : & T(1, l) \end{array}$$