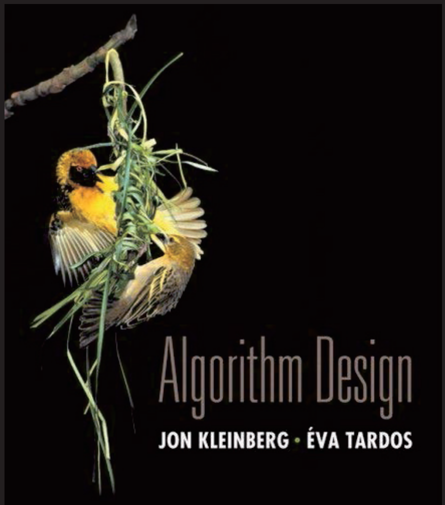



# Chapter 4

## Greedy Algorithms

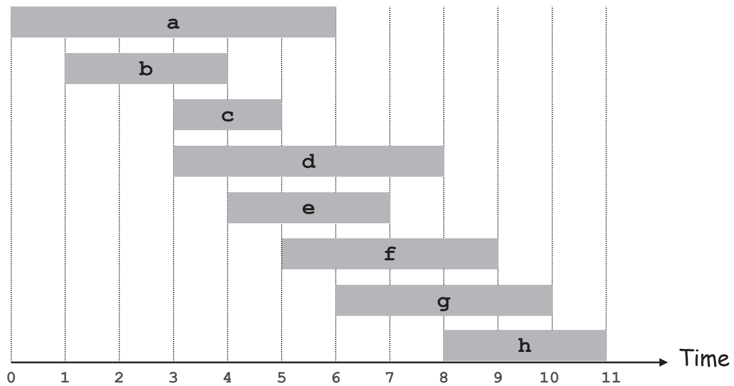


Acknowledgement: This lecture slide is revised and authorized from Prof. Kevin Wayne's Class
The original version and official versions are at http://www.cs.princeton.edu/~wayne/

# Interval Scheduling

## Interval Scheduling

Interval scheduling.
- Job j starts at $s_j$ and finishes at $f_j$.
- Two jobs compatible if they don't overlap.
- Goal: find maximum subset of mutually compatible jobs.






## Interval Scheduling: Greedy Algorithms

Greedy template. Consider jobs in some natural order.
Take each job provided it's compatible with the ones already taken.

- [Earliest start time] Consider jobs in ascending order of $s_j$.
- [Earliest finish time] Consider jobs in ascending order of $f_j$.
- [Shortest interval] Consider jobs in ascending order of $f_j - s_j$.
- [Fewest conflicts] For each job j, count the number of conflicting jobs $c_j$. Schedule in ascending order of $c_j$.

## Interval Scheduling: Greedy Algorithms

Greedy template. Consider jobs in some natural order.
Take each job provided it's compatible with the ones already taken.




## Interval Scheduling: Greedy Algorithm

Greedy algorithm. Consider jobs in increasing order of finish time.
Take each job provided it's compatible with the ones already taken.

```
Sort jobs by finish times so that f1 ≤ f2 ≤ ... ≤ fn.

  set of jobs selected
A ← φ
for j = 1 to n {
   if (job j compatible with A)
      A ← A ∪ {j}
}
return A
```

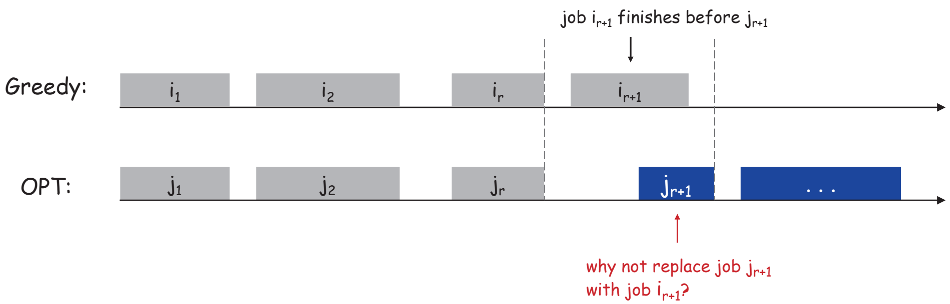
Implementation. O(n log n).
- Remember job j* that was added last to A.
- Job j is compatible with A if $s_j \geq f_{j^*}$.



## Interval Scheduling: Analysis

Theorem. Greedy algorithm is optimal.

Pf. (by contradiction)
- Assume greedy is not optimal, and let's see what happens.
- Let $i_1, i_2, \ldots i_k$ denote set of jobs selected by greedy.
- Let $j_1, j_2, \ldots j_m$ denote set of jobs in the optimal solution with $i_1 = j_1, i_2 = j_2, \ldots, i_r = j_r$ for the largest possible value of r.



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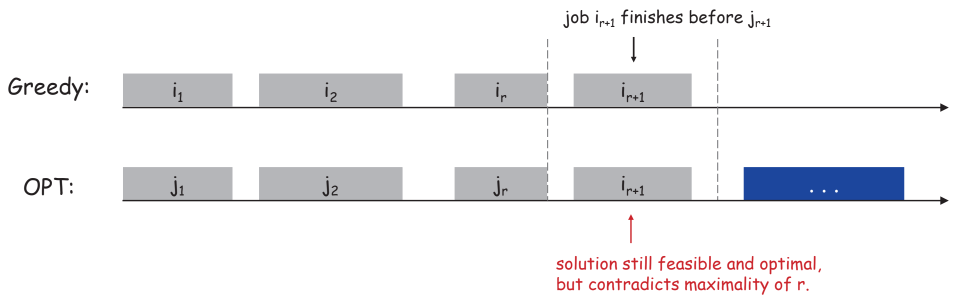


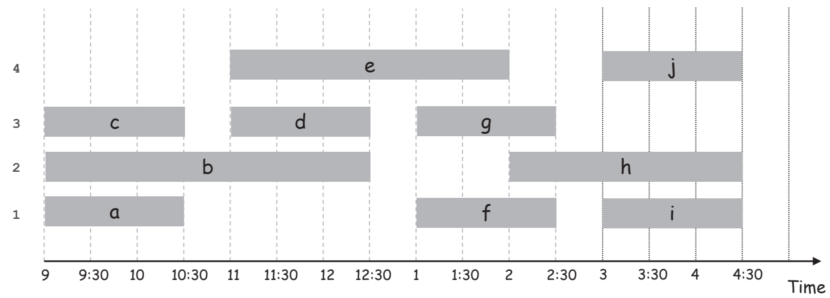
# Interval Partitioning



Interval partitioning.

- Lecture j starts at $s_j$ and finishes at $f_j$.
- Goal: find minimum number of classrooms to schedule all lectures so that no two occur at the same time in the same room.

Ex: This schedule uses 4 classrooms to schedule 10 lectures.



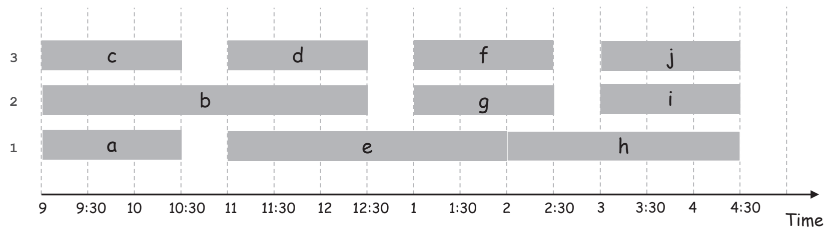





Interval partitioning.

- Lecture j starts at $s_j$ and finishes at $f_j$.
- Goal: find minimum number of classrooms to schedule all lectures so that no two occur at the same time in the same room.

Ex: This schedule uses only 3.







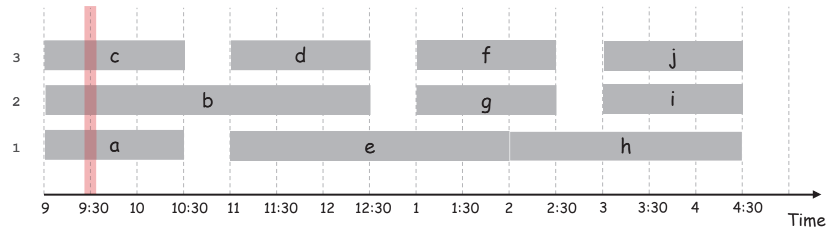

Def. The depth of a set of open intervals is the maximum number that contain any given time.

Key observation. Number of classrooms needed $\geq$ depth.

Ex: Depth of schedule below = 3 $\Rightarrow$ schedule below is optimal.

a, b, c all contain 9:30

Q. Does there always exist a schedule equal to depth of intervals?



## Interval Partitioning: Greedy Algorithm

Greedy algorithm. Consider lectures in increasing order of start time: assign lecture to any compatible classroom.

```
Sort intervals by starting time so that s1 ≤ s2 ≤ ... ≤ sn.
d ← 0  ← number of allocated classrooms

for j = 1 to n {
   if (lecture j is compatible with some classroom k)
      schedule lecture j in classroom k
   else
      allocate a new classroom d + 1
      schedule lecture j in classroom d + 1
      d ← d + 1
}
```

Implementation. O(n log n).
- For each classroom k, maintain the finish time of the last job added.
- Keep the classrooms in a priority queue.



## Interval Partitioning: Greedy Analysis

Observation. Greedy algorithm never schedules two incompatible lectures in the same classroom.

Theorem. Greedy algorithm is optimal.
Pf.
- Let d = number of classrooms that the greedy algorithm allocates.
- Classroom d is opened because we needed to schedule a job, say j, that is incompatible with all d-1 other classrooms.
- These d jobs each end after $s_j$.
- Since we sorted by start time, all these incompatibilities are caused by lectures that start no later than $s_j$.
- Thus, we have d lectures overlapping at time $s_j + \varepsilon$.
- Key observation $\Rightarrow$ all schedules use $\geq$ d classrooms. ▪



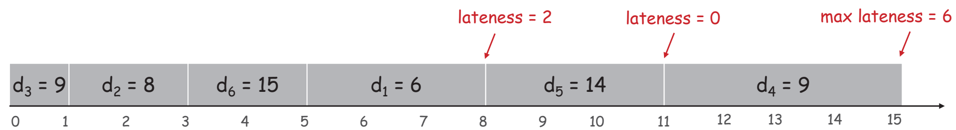
# Scheduling to Minimize Lateness

## Scheduling to Minimizing Lateness

Minimizing lateness problem.
- Single resource processes one job at a time.
- Job j requires $t_j$ units of processing time and is due at time $d_j$.
- If j starts at time $s_j$, it finishes at time $f_j = s_j + t_j$.
- Lateness: $\ell_j = \max\{0, f_j - d_j\}$.
- Goal: schedule all jobs to minimize maximum lateness $L = \max \ell_j$.

Ex:

| | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| $t_j$ | 3 | 2 | 1 | 4 | 3 | 2 |
| $d_j$ | 6 | 8 | 9 | 9 | 14 | 15 |



## Minimizing Lateness: Greedy Algorithms

Greedy template. Consider jobs in some order.

- [Shortest processing time first] Consider jobs in ascending order of processing time $t_j$.
- [Earliest deadline first] Consider jobs in ascending order of deadline $d_j$.
- [Smallest slack] Consider jobs in ascending order of slack $d_j - t_j$.



## Minimizing Lateness: Greedy Algorithms

Greedy template. Consider jobs in some order.

- [Shortest processing time first] Consider jobs in ascending order of processing time $t_j$.

| | 1 | 2 |
|---|---|---|
| $t_j$ | 1 | 10 |
| $d_j$ | 100 | 10 |

counterexample

- [Smallest slack] Consider jobs in ascending order of slack $d_j - t_j$.

| | 1 | 2 |
|---|---|---|
| $t_j$ | 1 | 10 |
| $d_j$ | 2 | 10 |

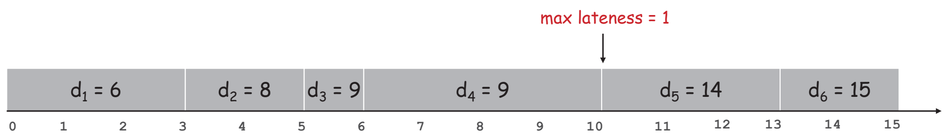
counterexample



## Minimizing Lateness: Greedy Algorithm

Greedy algorithm. Earliest deadline first.

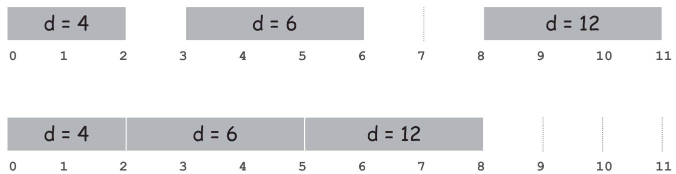
```
Sort n jobs by deadline so that d_1 ≤ d_2 ≤ … ≤ d_n

t ← 0
for j = 1 to n
   Assign job j to interval [t, t + t_j]
   s_j ← t, f_j ← t + t_j
   t ← t + t_j
output intervals [s_j, f_j]
```






## Minimizing Lateness: No Idle Time

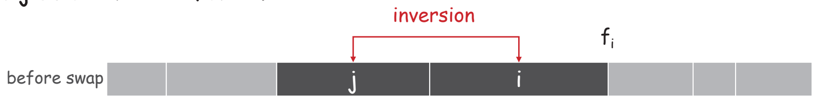
Observation. There exists an optimal schedule with no idle time.




Observation. The greedy schedule has no idle time.

## Minimizing Lateness: Inversions

Def. Given a schedule S, an inversion is a pair of jobs i and j such that: i < j but j scheduled before i.




[ as before, we assume jobs are numbered so that $d_1 \le d_2 \le \ldots \le d_n$ ]

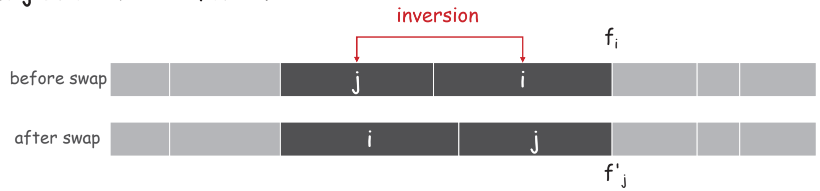
Observation. Greedy schedule has no inversions.

Observation. If a schedule (with no idle time) has an inversion, it has one with a pair of inverted jobs scheduled consecutively.



## Minimizing Lateness: Inversions

Def. Given a schedule S, an inversion is a pair of jobs i and j such that: i < j but j scheduled before i.




Claim. Swapping two consecutive, inverted jobs reduces the number of inversions by one and does not increase the max lateness.

Pf. Let $\ell$ be the lateness before the swap, and let $\ell'$ be it afterwards.

- $\ell'_k = \ell_k$ for all $k \ne i, j$
- $\ell'_i \le \ell_i$
- If job j is late:

$$\begin{aligned} \ell'_j &= f'_j - d_j && \text{(definition)} \\ &= f_i - d_j && (j \text{ finishes at time } f_i) \\ &\le f_i - d_i && (i < j) \\ &\le \ell_i && \text{(definition)} \end{aligned}$$



## Minimizing Lateness: Analysis of Greedy Algorithm

Theorem. Greedy schedule S is optimal.
Pf. Define S* to be an optimal schedule that has the fewest number of inversions, and let's see what happens.

- Can assume S* has no idle time.
- If S* has no inversions, then S = S*.
- If S* has an inversion, let i-j be an adjacent inversion.
  - swapping i and j does not increase the maximum lateness and strictly decreases the number of inversions
  - this contradicts definition of S* ▪



## Greedy Analysis Strategies

Greedy algorithm stays ahead. Show that after each step of the greedy algorithm, its solution is at least as good as any other algorithm's.

Structural. Discover a simple "structural" bound asserting that every possible solution must have a certain value. Then show that your algorithm always achieves this bound.

Exchange argument. Gradually transform any solution to the one found by the greedy algorithm without hurting its quality.

Other greedy algorithms. Kruskal, Prim, Dijkstra, Huffman, …

# Optimal Caching

## Optimal Offline Caching

Caching.
- Cache with capacity to store k items.
- Sequence of m item requests $d_1, d_2, \ldots, d_m$.
- Cache hit: item already in cache when requested.
- Cache miss: item not already in cache when requested: must bring requested item into cache, and evict some existing item, if full.

Goal. Eviction schedule that minimizes number of cache misses.

Ex: k = 2, initial cache = ab,
requests: a, b, c, b, c, a, a, b.
Optimal eviction schedule: 2 cache misses.

red = cache miss

| requests | cache | |
|---|---|---|
| a | a | b |
| b | a | b |
| c | c | b |
| b | c | b |
| c | c | b |
| a | a | b |
| a | a | b |
| b | a | b |



## Optimal Offline Caching: Farthest-In-Future

Farthest-in-future. Evict item in the cache that is not requested until farthest in the future.

current cache: a b c d e f

future queries: g a b c e d a b b a c d e a f a d e f g h ...

cache miss (g) — eject this one (f)

Theorem. [Bellady, 1960s] FF is optimal eviction schedule.
Pf. Algorithm and theorem are intuitive; proof is subtle.



## Reduced Eviction Schedules

Def. A reduced schedule is a schedule that only inserts an item into the cache in a step in which that item is requested.

Intuition. Can transform an unreduced schedule into a reduced one with no more cache misses.

| | | | |
|---|---|---|---|
| a | a | b | c |
| a | a | x | c |
| c | a | d | c |
| d | a | d | b |
| a | a | c | b |
| b | a | x | b |
| c | a | c | b |
| a | a | b | c |
| a | a | b | c |

an unreduced schedule

| | | | |
|---|---|---|---|
| a | a | b | c |
| a | a | b | c |
| c | a | b | c |
| d | a | d | c |
| a | a | d | c |
| b | a | d | b |
| c | a | c | b |
| a | a | c | b |
| a | a | c | b |

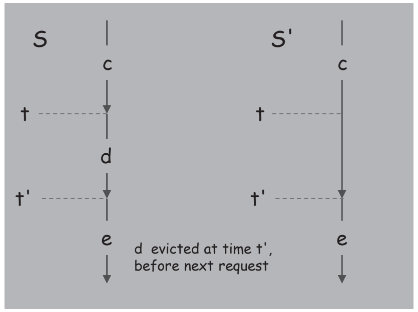
a reduced schedule

## Reduced Eviction Schedules

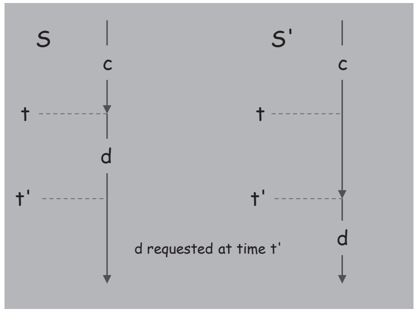
Claim. Given any unreduced schedule S, can transform it into a reduced schedule S' with no more cache misses.

Pf. (by induction on number of unreduced items) ← doesn't enter cache at requested time

- Suppose S brings d into the cache at time t, without a request.
- Let c be the item S evicts when it brings d into the cache.
- Case 1: d evicted at time t', before next request for d.
- Case 2: d requested at time t' before d is evicted. ▪




Case 1




Case 2



## Farthest-In-Future: Analysis

Theorem. FF is optimal eviction algorithm.

Pf. (by induction on number or requests j)

> Invariant: There exists an optimal reduced schedule S that makes the same eviction schedule as $S_{FF}$ through the first j+1 requests.

Let S be reduced schedule that satisfies invariant through j requests. We produce S' that satisfies invariant after j+1 requests.

- Consider $(j+1)^{st}$ request $d = d_{j+1}$.
- Since S and $S_{FF}$ have agreed up until now, they have the same cache contents before request j+1.
- Case 1: (d is already in the cache). S' = S satisfies invariant.
- Case 2: (d is not in the cache and S and $S_{FF}$ evict the same element). S' = S satisfies invariant.



## Farthest-In-Future: Analysis

Pf. (continued)

- Case 3: (d is not in the cache; $S_{FF}$ evicts e; S evicts f ≠ e).
  - begin construction of S' from S by evicting e instead of f

| | S | | | S' | | |
|---|---|---|---|---|---|---|
| j | same | e | f | same | e | f |
| j+1 | same | e | d | same | d | f |

  - now S' agrees with $S_{FF}$ on first j+1 requests; we show that having element f in cache is no worse than having element e



## Farthest-In-Future: Analysis

Let j' be the first time after j+1 that S and S' take a different action, and let g be item requested at time j'.

must involve e or f (or both)

| | S | | S' | |
|---|---|---|---|---|
| j' | same | e | same | f |

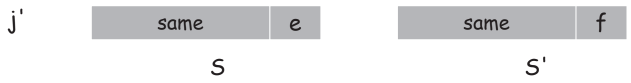
- Case 3a: g = e. Can't happen with Farthest-In-Future since there must be a request for f before e.
- Case 3b: g = f. Element f can't be in cache of S, so let e' be the element that S evicts.
  - if e' = e, S' accesses f from cache; now S and S' have same cache
  - if e' ≠ e, S' evicts e' and brings e into the cache; now S and S' have the same cache

Note: S' is no longer reduced, but can be transformed into a reduced schedule that agrees with $S_{FF}$ through step j+1

## Farthest-In-Future: Analysis

Let j' be the first time after j+1 that S and S' take a different action, and let g be item requested at time j'.

must involve e or f (or both)




otherwise S' would take the same action

- Case 3c: $g \neq e, f$. S must evict e.
  Make S' evict f; now S and S' have the same cache. ▪




## Caching Perspective

Online vs. offline algorithms.
- Offline: full sequence of requests is known a priori.
- Online (reality): requests are not known in advance.
- Caching is among most fundamental online problems in CS.

LIFO. Evict page brought in most recently.
LRU. Evict page whose most recent access was earliest.

FF with direction of time reversed!

Theorem. FF is optimal offline eviction algorithm.
- Provides basis for understanding and analyzing online algorithms.
- LRU is k-competitive. [Section 13.8]
- LIFO is arbitrarily bad.

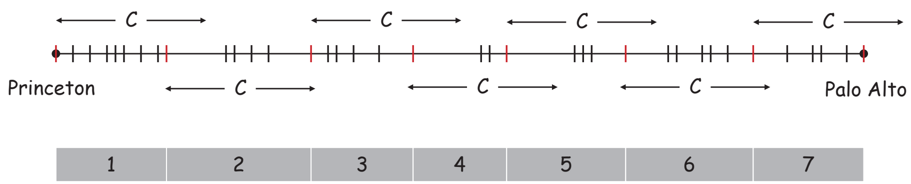


# Selecting Breakpoints

## Selecting Breakpoints

Selecting breakpoints.
- Road trip from Princeton to Palo Alto along fixed route.
- Refueling stations at certain points along the way.
- Fuel capacity = C.
- Goal: makes as few refueling stops as possible.

Greedy algorithm. Go as far as you can before refueling.



## Selecting Breakpoints: Greedy Algorithm

Truck driver's algorithm.

```
Sort breakpoints so that: 0 = b0 < b1 < b2 < ... < bn = L

S ← {0}    ← breakpoints selected
x ← 0      ← current location

while (x ≠ bn)
   let p be largest integer such that bp ≤ x + C
   if (bp = x)
      return "no solution"
   x ← bp
   S ← S ∪ {p}
return S
```

Implementation. O(n log n)

- Use binary search to select each breakpoint p.

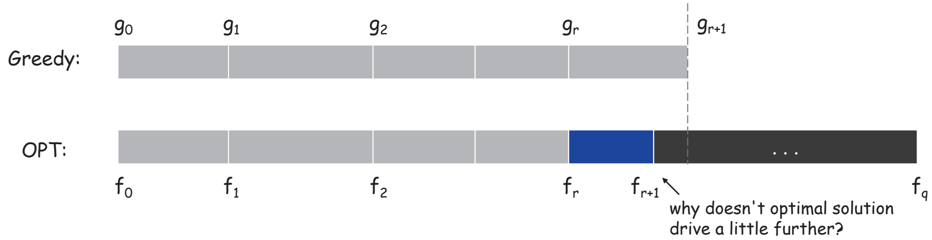


## Selecting Breakpoints: Correctness

Theorem. Greedy algorithm is optimal.

Pf. (by contradiction)

- Assume greedy is not optimal, and let's see what happens.
- Let $0 = g_0 < g_1 < \ldots < g_p = L$ denote set of breakpoints chosen by greedy.
- Let $0 = f_0 < f_1 < \ldots < f_q = L$ denote set of breakpoints in an optimal solution with $f_0 = g_0$, $f_1 = g_1$, . . . , $f_r = g_r$ for largest possible value of r.
- Note: $g_{r+1} > f_{r+1}$ by greedy choice of algorithm.



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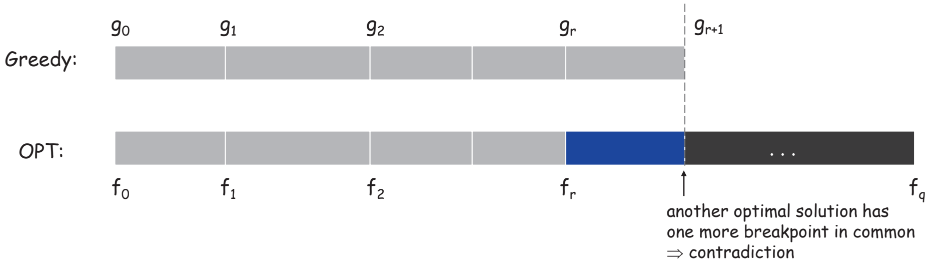





# Coin Changing

## Coin Changing

Goal. Given currency denominations: 1, 5, 10, 25, 100, devise a method to pay amount to customer using fewest number of coins.

Ex: 34¢.

Cashier's algorithm. At each iteration, add coin of the largest value that does not take us past the amount to be paid.

Ex: $2.89.



## Coin-Changing: Greedy Algorithm

Cashier's algorithm. At each iteration, add coin of the largest value that does not take us past the amount to be paid.

```
Sort coins denominations by value: c1 < c2 < … < cn.

  coins selected
S ← φ
while (x ≠ 0) {
   let k be largest integer such that ck ≤ x
   if (k = 0)
      return "no solution found"
   x ← x - ck
   S ← S ∪ {k}
}
return S
```

Q. Is cashier's algorithm optimal?



## Properties of optimal solution

| |
|---|
| penny=1<br>nickel=5<br>dime=10<br>quarter=25 |

Property. Number of pennies ≤ 4.
Pf. Replace 5 pennies with 1 nickel.

Property. Number of nickels ≤ 1.
Property. Number of quarters ≤ 3.

Property. Number of nickels + number of dimes ≤ 2.
Pf.
- Replace 3 dimes and 0 nickels with 1 quarter and 1 nickel;
- Replace 2 dimes and 1 nickel with 1 quarter.
- Recall: at most 1 nickel.



## Coin-Changing: Analysis of Greedy Algorithm

Theorem. Greedy algorithm is optimal for U.S. coinage: 1, 5, 10, 25, 100.
Pf. (by induction on x)
- Consider optimal way to change $c_k \le x < c_{k+1}$ : greedy takes coin k.
- We claim that any optimal solution must also take coin k.
  - if not, it needs enough coins of type $c_1, \ldots, c_{k-1}$ to add up to x
  - table below indicates no optimal solution can do this
- Problem reduces to coin-changing x - $c_k$ cents, which, by induction, is optimally solved by greedy algorithm. ▪

| k | $c_k$ | All optimal solutions must satisfy | Max value of coins 1, 2, …, k-1 in any OPT |
|---|---|---|---|
| 1 | 1 | P ≤ 4 | - |
| 2 | 5 | N ≤ 1 | 4 |
| 3 | 10 | N + D ≤ 2 | 4 + 5 = 9 |
| 4 | 25 | Q ≤ 3 | 20 + 4 = 24 |
| 5 | 100 | no limit | 75 + 24 = 99 |

## Is cashier's algorithm for any set of denominations?

Observation 1. Greedy algorithm is sub-optimal for US postal denominations: 1, 10, 21, 34, 70, 100, 350, 1225, 1500.

Counterexample. 140¢.
- Greedy: 100, 34, 1, 1, 1, 1, 1, 1.
- Optimal: 70, 70.

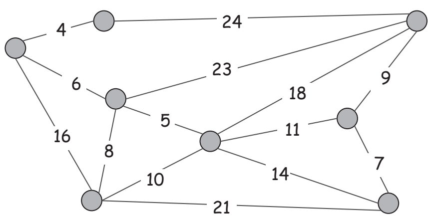
Observation 2. It may not even lead to a feasible solution if c1 > 1: 7, 8, 9.
- Cashier's algorithm: 15¢ = 9 + ???.
- Optimal: 15¢ = 7 + 8.



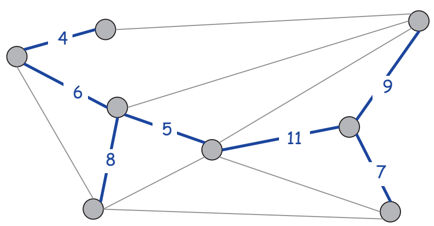
# Minimum Spanning Tree

## Minimum Spanning Tree

Minimum spanning tree. Given a connected graph G = (V, E) with real-valued edge weights $c_e$, an MST is a subset of the edges $T \subseteq E$ such that T is a spanning tree whose sum of edge weights is minimized.




G = (V, E)




T, $\Sigma_{e \in T} c_e = 50$

Cayley's Theorem. There are $n^{n-2}$ spanning trees of $K_n$.

can't solve by brute force



## Applications

MST is fundamental problem with diverse applications.

- Network design.
  - telephone, electrical, hydraulic, TV cable, computer, road
- Approximation algorithms for NP-hard problems.
  - traveling salesperson problem, Steiner tree
- Indirect applications.
  - max bottleneck paths
  - LDPC codes for error correction
  - image registration with Renyi entropy
  - learning salient features for real-time face verification
  - reducing data storage in sequencing amino acids in a protein
  - model locality of particle interactions in turbulent fluid flows
  - autoconfig protocol for Ethernet bridging to avoid cycles in a network
- Cluster analysis.

## Greedy Algorithms

Kruskal's algorithm. Start with T = $\phi$. Consider edges in ascending order of cost. Insert edge e in T unless doing so would create a cycle.

Reverse-Delete algorithm. Start with T = E. Consider edges in descending order of cost. Delete edge e from T unless doing so would disconnect T.

Prim's algorithm. Start with some root node s and greedily grow a tree T from s outward. At each step, add the cheapest edge e to T that has exactly one endpoint in T.

Remark. All three algorithms produce an MST.

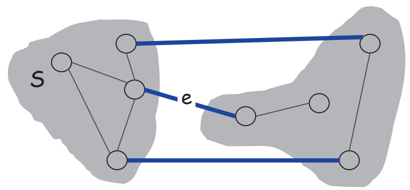


## Greedy Algorithms

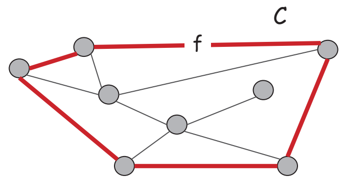
Simplifying assumption. All edge costs $c_e$ are distinct.

Cut property. Let S be any subset of nodes, and let e be the min cost edge with exactly one endpoint in S. Then the MST contains e.

Cycle property. Let C be any cycle, and let f be the max cost edge belonging to C. Then the MST does not contain f.




e is in the MST

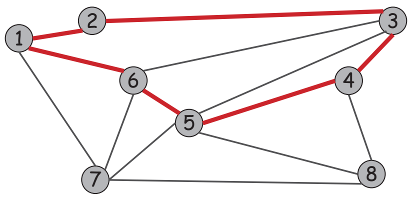



f is not in the MST



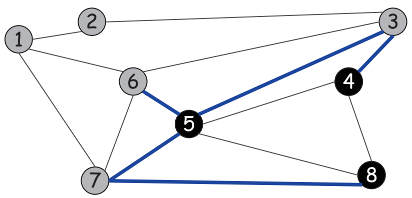
## Cycles and Cuts

Cycle. Set of edges the form a-b, b-c, c-d, …, y-z, z-a.



Cycle C = 1-2, 2-3, 3-4, 4-5, 5-6, 6-1

Cutset. A cut is a subset of nodes S. The corresponding cutset D is the subset of edges with exactly one endpoint in S.

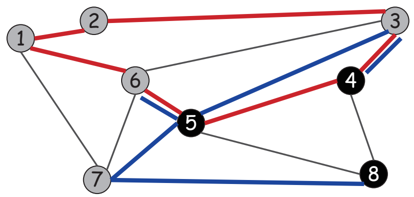


Cut S = { 4, 5, 8 }
Cutset D = 5-6, 5-7, 3-4, 3-5, 7-8



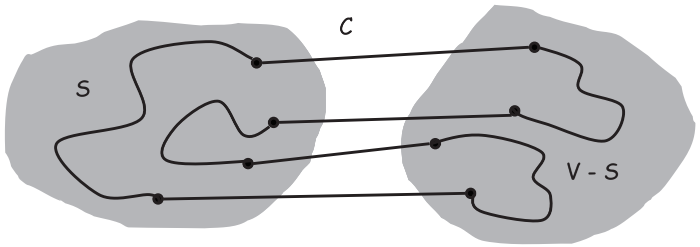
## Cycle-Cut Intersection

Claim. A cycle and a cutset intersect in an even number of edges.



Cycle C = 1-2, 2-3, 3-4, 4-5, 5-6, 6-1
Cutset D = 3-4, 3-5, 5-6, 5-7, 7-8
Intersection = 3-4, 5-6

Pf. (by picture)

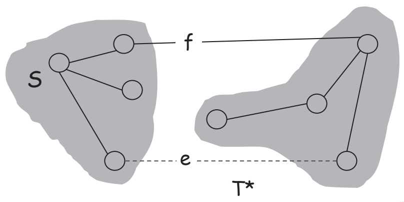


Simplifying assumption. All edge costs $c_e$ are distinct.

Cut property. Let S be any subset of nodes, and let e be the min cost edge with exactly one endpoint in S. Then the MST T* contains e.

Pf. (exchange argument)

- Suppose e does not belong to T*, and let's see what happens.
- Adding e to T* creates a cycle C in T*.
- Edge e is both in the cycle C and in the cutset D corresponding to S $\Rightarrow$ there exists another edge, say f, that is in both C and D.
- $T' = T^* \cup \{ e \} - \{ f \}$ is also a spanning tree.
- Since $c_e < c_f$, cost(T') < cost(T*).
- This is a contradiction. ▪





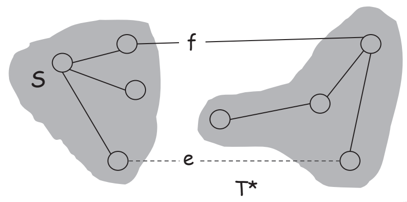



Simplifying assumption. All edge costs $c_e$ are distinct.

Cycle property. Let C be any cycle in G, and let f be the max cost edge belonging to C. Then the MST T* does not contain f.

Pf. (exchange argument)

- Suppose f belongs to T*, and let's see what happens.
- Deleting f from T* creates a cut S in T*.
- Edge f is both in the cycle C and in the cutset D corresponding to S $\Rightarrow$ there exists another edge, say e, that is in both C and D.
- $T' = T^* \cup \{ e \} - \{ f \}$ is also a spanning tree.
- Since $c_e < c_f$, cost(T') < cost(T*).
- This is a contradiction. ▪

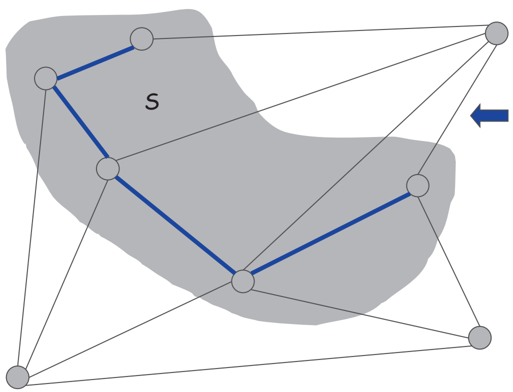





## Prim's Algorithm: Proof of Correctness

Prim's algorithm. [Jarník 1930, Dijkstra 1957, Prim 1959]

- Initialize S = any node.
- Apply cut property to S.
- Add min cost edge in cutset corresponding to S to T, and add one new explored node u to S.






## Implementation: Prim's Algorithm

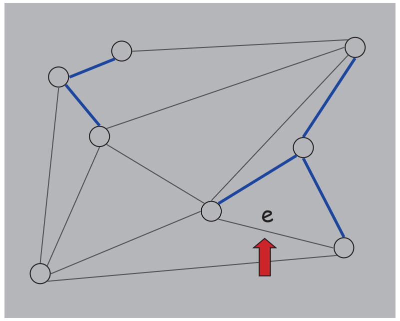
Implementation. Use a priority queue.

- Maintain set of explored nodes S.
- For each unexplored node v, maintain attachment cost a[v] = cost of cheapest edge v to a node in S.
- $O(n^2)$ with an array; $O(m \log n)$ with a binary heap.

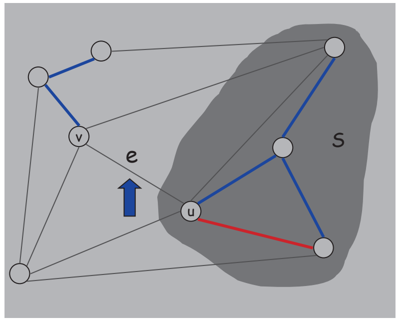
```
Prim(G, c) {
   foreach (v ∈ V) a[v] ← ∞
   Initialize an empty priority queue Q
   foreach (v ∈ V) insert v onto Q
   Initialize set of explored nodes S ← φ

   while (Q is not empty) {
      u ← delete min element from Q
      S ← S ∪ { u }
      foreach (edge e = (u, v) incident to u)
          if ((v ∉ S) and (c_e < a[v]))
             decrease priority a[v] to c_e
}
```

## Kruskal's Algorithm: Proof of Correctness

Kruskal's algorithm. [Kruskal, 1956]
- Consider edges in ascending order of weight.
- Case 1: If adding e to T creates a cycle, discard e according to cycle property.
- Case 2: Otherwise, insert e = (u, v) into T according to cut property where S = set of nodes in u's connected component.




Case 1




Case 2



## Implementation: Kruskal's Algorithm

Implementation. Use the union-find data structure.
- Build set T of edges in the MST.
- Maintain set for each connected component.
- O(m log n) for sorting and $O(m\ \underbrace{\alpha\ (m, n)}_{\text{essentially a constant}})$ for union-find.

$m \le n^2 \Rightarrow$ log m is O(log n)

```
Kruskal(G, c) {
   Sort edges weights so that c_1 ≤ c_2 ≤ ... ≤ c_m.
   T ← φ

   foreach (u ∈ V) make a set containing singleton u

   for i = 1 to m          are u and v in different connected components?
      (u,v) = e_i
      if (u and v are in different sets) {
         T ← T ∪ {e_i}
         merge the sets containing u and v
      }                    merge two components
   return T
}
```



## Lexicographic Tiebreaking

To remove the assumption that all edge costs are distinct: perturb all edge costs by tiny amounts to break any ties.

Impact. Kruskal and Prim only interact with costs via pairwise comparisons. If perturbations are sufficiently small, MST with perturbed costs is MST with original costs.

e.g., if all edge costs are integers, perturbing cost of edge $e_i$ by i / $n^2$

Implementation. Can handle arbitrarily small perturbations implicitly by breaking ties lexicographically, according to index.

```
boolean less(i, j) {
   if      (cost(e_i) < cost(e_j)) return true
   else if (cost(e_i) > cost(e_j)) return false
   else if (i < j)                 return true
   else                            return false
}
```



## MST Algorithms: Theory

Deterministic comparison based algorithms.
- O(m log n) [Jarník, Prim, Dijkstra, Kruskal, Boruvka]
- O(m log log n). [Cheriton-Tarjan 1976, Yao 1975]
- O(m $\beta$(m, n)). [Fredman-Tarjan 1987]
- O(m log $\beta$(m, n)). [Gabow-Galil-Spencer-Tarjan 1986]
- O(m $\alpha$ (m, n)). [Chazelle 2000]

Holy grail. O(m).

Notable.
- O(m) randomized. [Karger-Klein-Tarjan 1995]
- O(m) verification. [Dixon-Rauch-Tarjan 1992]

Euclidean.
- 2-d: O(n log n). compute MST of edges in Delaunay
- k-d: O(k $n^2$). dense Prim

# Clustering

## Clustering

Clustering. Given a set U of n objects labeled $p_1, \ldots, p_n$, classify into coherent groups.

photos, documents. micro-organisms

Distance function. Numeric value specifying "closeness" of two objects.

number of corresponding pixels whose intensities differ by some threshold

Fundamental problem. Divide into clusters so that points in different clusters are far apart.

- Routing in mobile ad hoc networks.
- Identify patterns in gene expression.
- Document categorization for web search.
- Similarity searching in medical image databases
- Skycat: cluster $10^9$ sky objects into stars, quasars, galaxies.



## Clustering of Maximum Spacing

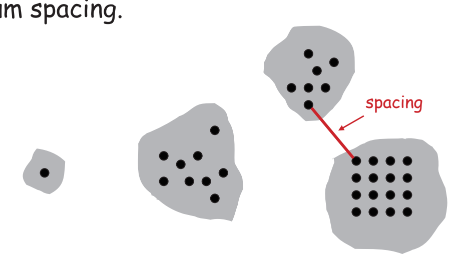
k-clustering. Divide objects into k non-empty groups.

Distance function. Assume it satisfies several natural properties.

- $d(p_i, p_j) = 0$ iff $p_i = p_j$ (identity of indiscernibles)
- $d(p_i, p_j) \geq 0$ (nonnegativity)
- $d(p_i, p_j) = d(p_j, p_i)$ (symmetry)

Spacing. Min distance between any pair of points in different clusters.

Clustering of maximum spacing. Given an integer k, find a k-clustering of maximum spacing.




k = 4



## Greedy Clustering Algorithm

Single-link k-clustering algorithm.

- Form a graph on the vertex set U, corresponding to n clusters.
- Find the closest pair of objects such that each object is in a different cluster, and add an edge between them.
- Repeat n-k times until there are exactly k clusters.

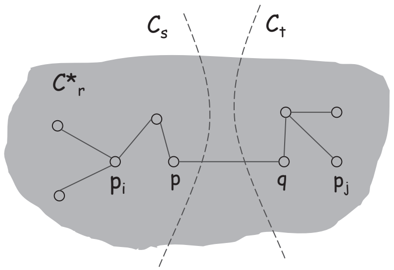
Key observation. This procedure is precisely Kruskal's algorithm (except we stop when there are k connected components).

Remark. Equivalent to finding an MST and deleting the k-1 most expensive edges.

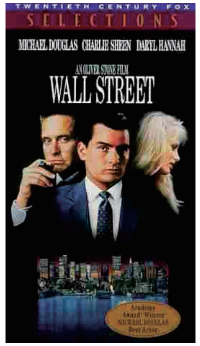
# Greedy Clustering Algorithm: Analysis

Theorem. Let $C^*$ denote the clustering $C^*_1, \ldots, C^*_k$ formed by deleting the k-1 most expensive edges of a MST. $C^*$ is a k-clustering of max spacing.

Pf. Let $C$ denote some other clustering $C_1, \ldots, C_k$.
- The spacing of $C^*$ is the length $d^*$ of the $(k-1)^{st}$ most expensive edge.
- Let $p_i$, $p_j$ be in the same cluster in $C^*$, say $C^*_r$, but different clusters in $C$, say $C_s$ and $C_t$.
- Some edge $(p, q)$ on $p_i$-$p_j$ path in $C^*_r$ spans two different clusters in $C$.
- All edges on $p_i$-$p_j$ path have length $\leq d^*$ since Kruskal chose them.
- Spacing of $C$ is $\leq d^*$ since p and q are in different clusters. ▪






Greed is good.

Greed is right.

Greed works.

Greed clarifies, cuts through, and captures the essence of the evolutionary spirit.

*- Gordon Gecko (Michael Douglas)*