# Percolation Degree of Secondary Users in Cognitive Networks

Luoyi Fu, Liang Qian, Xiaohua Tian, Huan Tang, Ning Liu, Guanglin Zhang, Xinbing Wang

***Abstract*—A cognitive network refers to the one where two overlaid structures, called primary and secondary networks coexist. The primary network consists of primary nodes who are licensed spectrum users while the secondary network comprises unauthorized users that have to access the licensed spectrum opportunistically. In this paper, we study the percolation degree of the secondary network to achieve $k$-percolation in large scale cognitive radio networks. The percolation degree is defined as the number of nearest neighbors for each secondary user when there are at least $k$ vertex-disjoint paths existing between any two secondary relays in the percolated cluster. The percolated cluster is formed when there are an infinite number of mutually connected secondary users spanning the whole network. Each user in the cluster is possibly connected to several neighbors, inducing more communication links between any two of them. Since nodes located near the boundary have fewer neighbors, the boundary effect becomes a bottleneck in determining the percolation degree. For cognitive networks, when the primary node density becomes considerably large, the boundary effect spreads inside the network. The transmission area of most secondary users who are located near the primary nodes decreases due to the restriction of the primary network. Therefore, to ensure $k$-connectivity in the percolated cluster, each secondary user must be connected to more neighbors, and the percolation degree of the secondary network yields a function of the primary node density. We specify the relationship into three regimes regarding the topology variation of the cognitive network. A closed-form expression of the percolation degree under different primary node densities is presented. The expression characterizes the connectivity strength in the secondary percolated cluster, therefore providing analytical insight on fault tolerance improvement in cognitive networks.**

***Index Terms*—Connectivity, Continuum Percolation, Scaling Law**

## I. Introduction

THE INCREASING demand for available communication spectrum has drawn intense attention in many fields, such as mobile communication, military detection and environmental monitoring. An appealing solution is re-exploration of the underutilized licensed spectrum. The strategy introduced in [1] allows unlicensed users to take advantage of the temporarily unoccupied spectrum on the condition that they cannot cause destructive interferences to licensed users. By adopting a hierarchical access control, the spectrum efficiency and reusability can be greatly improved.

The cognitive network, which has spawn tremendous interest in recent studies on wireless communication, is a new communication paradigm to meet the spectrum demand aforementioned. In cognitive networks proposed in [3], there are two overlaid networks called the primary network and the secondary network, respectively. The primary network is composed of licensed users, who transmit based on their own protocols and transmission schemes, independent of the secondary users. In contrast, secondary users form unlicensed community who can only transmit when the communication channels are idle from primary users. They have to identify and explore instantaneous and local spectrum opportunities to avoid affecting the communication of the primary network. Therefore, secondary users are equipped with cognitive radio so that they can detect the spectrum environment and adjust their communication operations accordingly. The primary and secondary networks are independent of each other in terms of node distribution, but can transmit over the same time, space and frequency. Different from homogeneous networks where all nodes have equal transmission opportunities, cognitive networks are heterogeneous in terms of user priorities.

Though the cognitive network addresses critical concerns in spectrum utilization, the analysis of the heterogeneous network structure brings greater challenges to researchers sharing an interest in this field. Among the fundamental limits concerning the communication network, connectivity has been discussed intensively as one of the most basic issues. When two users communicate with each other, they must be connected firstly. The information can be disseminated over the whole network area when all the users in the network are fully connected as a huge cluster. In cognitive networks, secondary users communicate with each other based on the information they detect in the cognitive environment. Consequently, connectivity of the secondary network is influenced by either of the network topology and can therefore reflect the interaction between the two networks.

Investigating the connectivity of the secondary network can lead to a better understanding of the cognitive network paradigm. Due to the subordinative status, secondary users suffer from impulsive destruction on the communication links from the primary network, which imposes fault tolerance in the secondary network on an urgent issue. Intuitively, the network with higher connectivity strength is more resistant to fault emergencies. When there are several paths between any two communicating objects, the loss caused by failure on one path can be minimized by increasing communication opportunities on alternative paths. Thus, not only connectivity of secondary network is worth of analysis, a concrete metric to measure the connectivity strength of the secondary network is also necessary for us to design better cognitive networks with high

Manuscript received 31 December 2011; revised 14 May 2012.
The authors are with the Department of Electronic Engineering, Shanghai Jiao Tong University, Shanghai 200240, China (e-mail:{yiluofu,lqian,xtian,htang,jsrdln,glzhang,xwang8}@sjtu.edu.cn).
Digital Object Identifier 10.1109/JSAC.2012.1211xx.

fault tolerance of secondary users. Most previous works, e.g., [10], [16], investigate the connectivity strength with all nodes involved in the network. However, the connectivity strength concerning the percolation phenomenon, which is previously studied as a phase transition of the immediate formation of an infinite cluster, is still an open issue.

In this paper, we focus on the connectivity characteristics of the secondary network. Specifically, we investigate the scenario of percolation where there are an infinite number of mutually connected secondary users spanning the whole network area, called the percolated cluster. Our work derives the percolation degree in the scenario, defined as the nearest number of neighbors for the secondary network so that there are at least $k$ vertex-disjoint communication paths between any two secondary users in the percolated cluster.

To figure out the percolation degree, we first explore the condition under which there is a positive probability for the secondary network to achieve percolation. The condition specifies the requirements on network parameters concerning the percolation phenomenon of secondary users. Under the condition, we derive the lower bound of the percolation degree $l^*$. When each secondary user is connected to its $l^*$ nearest neighbors, it occurs with positive probability for the secondary network to achieve $k$-percolation, namely the secondary percolated cluster is $k$-connected. The lower bound is asymptotically achievable when the primary node density is sufficiently low. The impact of the primary topology on the secondary communication links becomes apparent with the increase of the primary node density. As the cognitive network exhibits different topologies with respect to different primary node densities, the expression of the percolation degree can be divided into three regimes. Our work establishes a full relationship between the percolation degree and the primary node density, which characterizes connectivity of the secondary network as an illustration of properties of the cognitive network paradigm.

The remainder of the paper is organized as follows. In Section II, we give an overview of related works on network connectivity and percolation theory. Section III presents our network model and the specific definition of percolation degree is given. Section IV lists the main results of our work whereas Section V briefly introduces the solution of the main results. We investigate the scenario of percolation in Section VI and derive the critical primary node density regarding percolation in the secondary network. Section VII is contributed for deriving both the lower and upper bounds of the percolation degree as well as a closed-form expression of the relationship between percolation degree and the density of primary nodes. We discuss some prospective extensions in Section VIII and give concluding remarks in Section IX.

## II. Related Works

Initiated by the seminal work by Gupta and Kumar in [9], where they discuss the critical power for asymptotic full connectivity, connectivity has been under intensive study over the last decades. When two nodes communicate with each other, they must be connected firstly, namely they are within each other's transmission range. As to the whole network, packets can be relayed between any given pair of nodes successfully when the whole network is connected within a big cluster. In [9], it is pointed out that to achieve asymptotic connectivity, the critical transmission range should be $O(\sqrt{\log n/n})$ for a dense network with $n$ nodes randomly scattered in a unit-area square.

Later on, Dousse *et al.* introduce percolation theory into their work [16], where they relax the condition of full connectivity to the less restrictive level that the network is connected within a set of nodes spanning the entire network plane. The set is specified as a cluster composed of an infinite number of nodes and there exists at least one communication link between any two of them. The percolation theory has been adopted in a flurry of research in wireless networks and demonstrated to be a useful mathematical tool in solving various network problems. In [5], Franceschetti *et al.* show that the capacity of wireless networks can be greatly improved via the establishment of percolation highway, which is a big breakthrough of the pioneering work from Gupta and kumar [17]. Following that, the percolation phenomenon of wireless networks arouses increasing interest in the study of network characteristics. Then connectivity and delay tradeoff in cognitive networks is investigated by Ren and Zhao in [10]. The continuum percolation theory is utilized to study the connectivity and coverage in three dimensional network in [11]. And tradeoff between the number of neighbors and spectrum opportunities for secondary users in the cognitive network is revealed in [19]. Although helpful for gaining a better understanding of percolation phenomenon in communication networks, the major attention of all these works is limited to the scenario where the phase transition of percolation occurs in the network. It still remains unclear what are the specific properties of connectivity of the percolation cluster, with respect to the network topology.

Although one-connectivity has been taken into account in most related work, it only suffices to guarantee the network communication. To meet the demand of high fault tolerance of wireless communication, a class of large wireless systems, such as ad hoc network, mesh network and sensor network, has emerged for analysis of $k$-connectivity properties in wireless networks, which requires that the number of communication links between any two users in the network is at least $k$. In [2], Li *et al.* are the first to give loose lower and upper bounds of the critical transmission range for $k$-connectivity in a dense network. Then Wan and Yi show in [14] that the asymptotically critical transmission range of $k$-connectivity in the homogeneous network is $r_n = \Omega(\sqrt{\frac{\log n+(2k-1)\log\log n}{\pi n}})$. Their work is based on the result Penrose presents in [13] that the minimum transmission range for a network to achieve $k$-connectivity is the same as that when the network has minimum node degree $k$. Following that, the phase transition width of $k$-connectivity in wireless networks is characterized in [18]. And the node deployment pattern for $k$-connectivity in three dimensional is proposed in [15]. As an important metric, connectivity is also taken into consideration in some real applications such as sensor networks [20] [21], P2P networks [22] and VANETs [23]. However, none of these works are studied in the context of the cognitive network, where the communication is complicated due to the coexistence of both the primary and secondary networks. Therefore, in cognitive networks, the analysis of $k$-connectivity of the secondary network must take into consideration the impact of the primary network. In this paper, we will investigate the $k$-connectivity of the percolation cluster in the secondary network and provide analytical result of the percolation degree.

## III. System Model

### A. Network Topology

To study the asymptotic behavior of scaling characteristics of the network, we model the network topology in two cases, namely the dense network and the extended one. In a dense network, nodes are assumed to be located within a unit-area region with node density of $n$ that can go to infinity. In an extended network, the network area is assumed to be $\sqrt{n}\times\sqrt{n}$ while the node density is a constant (usually normalized to unit value). We will first focus our problem in a dense network.

In dense network model, we assume the primary and secondary network coexist in a two dimensional unit-area square denoted by $\mathcal{S}$. And we adopt the static Poisson Boolean model in [4] to characterize the node distribution. The locations of the nodes follow a Poisson Point Process with density $\lambda$, i.e., nodes are uniformly distributed in the network area and the number of the nodes is a random variable of Poisson distribution. Each node is represented by a disk centered at the location of the node with radius $r/2$, where $r$ denotes the transmission range independent of $\lambda$. Moreover, we denote by $B(\lambda, r)$ as the disk model.

In two dimensional Poisson Boolean model $B(\lambda, r)$, two nodes are connected if their disks overlap. They are considered to be disconnected once the distance between them is larger than $r$. Thus the transmission radius is a critical parameter in determining the connectivity of the proposed boolean model. The whole network is connected together when there is at least one chain of connected disks between any two nodes. Note that in the boolean model the transmission range for each node is twice the radius of the disks. This is because two nodes can communicate successfully given that they are located within each other's transmission range and the distance between two connected disks is no larger than $r$. We focus our study on the case where all the nodes communicate with the same transmission range in the network. Note that the case of heterogeneous transmission radiuses is beyond the scope of this paper and can be found in a flurry of related works. In the following we will describe the two overlaid networks respectively. As an illustration, Fig. 1 shows the network topology of the cognitive radio (CR) network.

*1) Primary Network:* As the primary network consists of licensed users who operate independently of secondary users, analysis of the primary network characteristics is similar to that of the homogeneous cases. In our discussion, we assume that the primary node distribution follows a Poisson point process with density $n$, and the transmission range for each primary user is $r_p$.

*2) Secondary network:* Similarly, secondary nodes are deployed in the network as a two dimensional Poisson point process with density $m$, which is independent of the primary network. In this paper, we are mainly interested in the asymptotic characteristics of the network. That is, we focus on the events that occur with probability approaching 1 when $m$ goes to infinity. Actually, we denote by $\zeta(m)$ as the density of active secondary users, who are located in the vacant space of the primary rejection disks that prohibit the secondary users inside from transmission, as will be defined later. Here $\zeta(m)$ is obtained by dividing the number of active secondary users to the whole area of the total network. We assume that all the secondary users employ the same transmission range $r_s$. In the CR network, secondary users possess the cognitive ability to sense the communication environment, which enables them to switch transmission status between on and off according to the primary network. Moreover, it is commonly assumed that the number of secondary nodes are much larger than that of primary ones so that the secondary nodes have available opportunistic spectrum access (OSA).

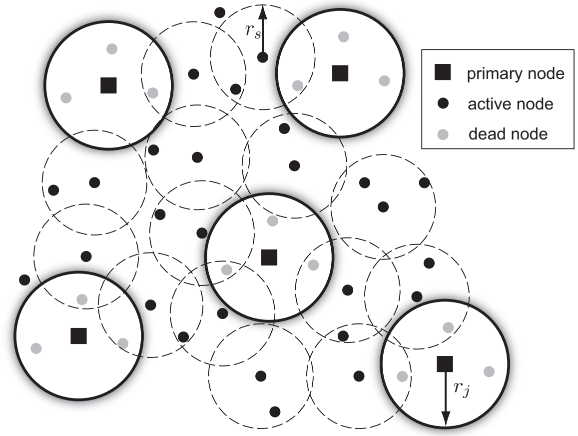


Fig. 1. Topology of CR Network. Primary users have a rejection radius $r_j$, secondary users in this region are not allowed to transmit information.

*3) Primary rejection region:* Restricted by the higher priority of the primary network, secondary users selectively transmit to ensure their existence undetected by the primary users. In our work, we specify this limitation by defining a rejection region for each primary user, given as follows.

*Definition 1:* Denote $r_j$ as the rejection radius and assume that $r_j = \eta r_s$ $(\eta > 1)$. A disk centered at a primary node with a radius $r_j$ is called the rejection region (RR). Any secondary user inside this region is enforced to turn off and thus not allowed to transmit.

Notice that the rejection radius is related to the secondary transmission radius with a proportional parameter larger than one. The mechanism behind is that the rejection region is introduced in order to prevent the communication activities of secondary users from generating negative effect on primary users. Thus the size of the rejection region is determined by the transmission power, which can be reflected by the transmission radius of secondary users. Furthermore, the rejection radius should be larger than the transmission radius to ensure that secondary transmitted signal is feebly negligible in the area where primary users are located. In other words, this helps to get us out of taking into account the interference caused by the primary users to the secondary ones, leaving the results unaffected in order sense.

Due to the spectrum limitation, all the secondary users are at the distance of at least $r_j$ away from primary users, mitigating the severe interference the primary network suffers from the secondary one. With primary nodes scattered randomly over the unit-area square, the whole secondary network is split into two parts, the *active nodes* located outside RR of any primary users and the *dead nodes* inside RR of at least one primary user. Note that dead nodes are turned off compulsively and therefore cannot communicate with neighboring nodes. Only active nodes can serve as packet relays in the secondary network.

### B. Percolation Degree

The range assignment in the Poisson Boolean model characterizes connectivity requirements in the network with regards to the transmission radius. A group of connected nodes is defined as a cluster, denoted by $C_i(\lambda, r)$. Let $N_i(\lambda, r)$ represent the number of mutually connected nodes in the cluster. We will use the main result of the continuum percolation theory in [8]. In a Poisson Boolean model $B(\lambda, r)$, if $\lambda r^2 > p_c$, then the probability that there exists an infinite cluster in the network approaches 1. $\lambda r^2$ is called the percolation probability of two dimensional network and $p_c$ is the critical value in the two dimensional Poisson Boolean model. Note that $p_c$ does not need to be specifically formulated, and it is shown in [12] that the analytical result of $p_c$ is in the range of $(0.7698, 3.372)$.

Recall that only active nodes can transmit information in the secondary network. Thus the secondary network is percolated when there exists an infinite cluster among the active nodes. Our work is concerned with the minimum number of vertex disjoint paths between any two active nodes in the infinite cluster. To specify this problem, we introduce the percolation degree defined below.

*Definition 2:* The secondary network is percolated when there exists a secondary cluster $\mathcal{C}(m, r_s)$ such that the number of secondary nodes in $\mathcal{C}(m, r_s)$, denoted by $N(\mathcal{C}(m, r_s))$, goes to infinity. If $\mathcal{C}(m, r_s)$ is $k$-connected, *i.e.*, for each pair of nodes in $\mathcal{C}(m, r_s)$ there are at least $k$ vertex disjoint paths connecting them, then the network is $k$-percolated. And the percolation degree is defined as the minimum number of neighbors[1] for each secondary user in $k$-percolated network.

The definition investigates the nearest neighbor number with regards to the connectivity strength of the percolation cluster in the secondary network. While the percolation demonstrates a phase transition phenomenon where the network becomes connected over the whole area once the percolation probability is above the threshold, the situation of multiple communication links between any two connected nodes in the percolation cluster is studied. Consequently, we will be able to investigate the connectivity properties inside the percolation cluster composed of an infinite number of nodes spanning the whole network area. In this work, we will first investigate the condition when there exists a huge connected component in the secondary network, and then will concentrate on deriving percolation degree in different network topologies.

[1]Note that communication can occur only between two SUs. Therefore the neighbors of a secondary user represent those secondary users who are nearest to it in terms of distance.

## IV. Main Results

Our work studies the percolation behavior in the secondary network and specifies the percolation degree of secondary users corresponding to different cognitive network topologies. The main results are given as follows:

1) For the secondary network, there exists a lower bound of the percolation degree $l^*$, defined as the minimum number of nearest neighbors required for each secondary user. There exists a primary network with node density $n < n^* = p_c/[r_s^2\sqrt{\eta^2 - \frac{1}{4}}]$, such that when each secondary user is connected to its $l^*$ nearest neighbors, the induced secondary network is $k$-percolated, namely there are at least $k$ communication links between any secondary users in the percolated cluster. And we find that

$$l^* = e\rho^2[\log\zeta(m) + k\log\log\zeta(m)],$$

where it is satisfied that $\lim_{m\to\infty}\zeta(m) \to \infty$ and $\rho$ is a constant larger than 1.

2) In cognitive networks, the percolation degree is a function of the primary node density $n$. The minimum neighbor number $l$ required for each secondary user follows that

$$l = \begin{cases} \frac{e\rho^2}{\delta}[\log\zeta(m) + k\log\log\zeta(m)] & , n \in (\frac{\eta^2-\frac{1}{4}}{(\eta+1)^2}n^*, n^*) \\ e\rho^2[\log\zeta(m) + k\log\log\zeta(m)] & , n \in (0, \frac{\eta^2-\frac{1}{4}}{(\eta+1)^2}n^*) \end{cases}$$

Here $\delta$ is a variable with regards to $n$.

The lower bound of the percolation degree is achievable under the optimal primary network topology, where the restriction of primary network on the secondary network connectivity can be asymptotically neglected. Thus, the percolation degree is mainly determined by active nodes serving as relays in the secondary network. In the cases where the primary users are sparsely deployed all over the network area, the lower bound can be achieved with a positive probability. The primary boundary effect becomes dominant as the number of primary users increases. The percolation degree varies more quickly with higher primary node density due to the rapid contraction of secondary transmission range. The main proof of our results is given in Section 7.

## V. Overview of the Solution

Our problem is defined in the cognitive radio network and we focus on the percolation behavior of the secondary network. To allow the formation of huge secondary cluster, we must firstly solve the problem concerning the requirements on the primary network topology that will lead to percolation occurrence among secondary users. It is required in the solution that there exists infinite vacant space in the network area. The vacant space visualizes the area in the network that is not covered by the rejection region. And we call the vacant space is infinite when its area is ineligible compared to the whole network area under the limitation of $m$. Secondary users can operate to communicate in this area when the spectrum is unoccupied by the primary users. Those secondary users which are connected in the huge cluster must be located in the vacant space. So when there is an infinite vacant space in the network, there is a positive probability that percolation can be achieved in the secondary network.

To derive the percolation degree of the secondary network, we consider the connectivity characteristics regarding the active secondary users. In Penrose's work [13], it is demonstrated that there exists a strong relationship between the average node degree and the connectivity degree of the whole network. When the minimum node degree for all the nodes in the network is above $k$, the network becomes $k$-connected immediately. Hence, we can focus our analysis on the average node degree of the active secondary users. In [14], Wan et al. study the the $k-$connectivity from the perspective of asymptotic transmission radius in wireless ad hoc networks, showing that the boundary effect is a critical factor in restricting the average node degree over all nodes in the network. Under the cognitive network model present in this paper, the boundary effect has been extended to the border of the rejection region where neighbors of active secondary users are located. As a result, the number of useful neighbors with which these active users can communicate decreases, making the average node degree yield a function of the area of the border between the primary and secondary networks. We will specify the network topologies in different scenarios and thus derive the border area to enable discussion on the average node degree of secondary users.

The main difficulty of this paper lies in that we need to consider the interaction of two overlaid networks coexisting within the same area. In cognitive networks, the interaction is specified by the rejection region where secondary users are not allowed for communication. The rejection region is directly determined by the location of primary users and the transmission power of secondary users. We should focus on their interplay and investigate its cooresponding impact on network performance. Note that the characteristic is different from most related works on connectivity analysis conducted in homogeneous networks. Utilizing the result, we can obtain more insights of heterogenous network models composed of different user communities.

## VI. Critical Primary Node Density

From the definition of percolation degree, the secondary network is percolated when there exists an infinite active cluster. As the active cluster must exist outside the primary rejection region, the distribution of active nodes is influenced by the primary network topology. Consider the primary network, primary nodes are uniformly distributed over the whole network area according to a poisson point process with node density $n$. Hence the number of primary users located in a given area is a random variable following a poisson distribution with parameter $n$ multiplied by the size of the area. In this section, we will study the critical primary node density $n^*$ above which the secondary network can be percolated.

### *A. Vacant Space*

As is specified in the network topology, the secondary users are not allowed to transmit in the rejection region centered at primary nodes. When the primary node density gets higher, the number of active nodes decreases. And there exists a critical primary node density $n^*$ above which secondary network is broken into several clusters of finite size, i.e., the number of nodes in any secondary cluster is finite. The percolation is unachievable in this scenario since there is no infinite vacant space in the primary network where the percolated cluster can exist. When the primary node density are smaller than $n^*$, there is a positive probability that a huge cluster composed of an infinite number of secondary nodes exists. The following theorem characterizes the value of $n^*$.

*Theorem 1:* In the cognitive radio network where $N_i(m, r_s)$ represents the number of nodes in the secondary cluster $C_i(m, r_s)$, the critical primary node density $n^*$ satisfies that $n^* r_s^2(\eta^2 - \frac{1}{4}) = p_c$, provided that

$$\mathbf{Pr}(\sup\{N_i(m, r_s) \mid n < n^*\} = \infty) > 0,$$

*Proof:* The proof of the theorem is divided into two parts. Notice that $n^*$ is the critical percolation density in the poisson boolean model $B(n, \sqrt{r_j^2 - (\frac{r_s}{2})^2})$. In the first part, we will show that when there exists an infinite vacant space in $B(n, \sqrt{r_j^2 - (\frac{r_s}{2})^2})$, the secondary network is percolated with a positive probability. Then the second part completes the proof that when the primary poisson boolean model with density $n_*$ is not percolated, there exists an infinite vacant space in the network area where the secondary percolation cluster can exist.

From the condition of the active secondary users, we can know that the secondary nodes connected in the percolation cluster must be located outside the rejection disk of any primary node. When there is an infinite vacant space outside the primary rejection disks with radius $r_j$, the number of active secondary users goes to infinity. Hence, as long as the connectivity of active secondary nodes is guaranteed, percolation in the secondary network can be achieved. A natural deduction from the discussion above is that when the percolation probability for the rejection disk is less than the critical percolation probability of the two dimensional network, the secondary network can be percolated. However, the conclusion can be developed even further.

Consider a secondary communication path in the percolation cluster depicted in Fig. 2, any secondary node along this path must be at the distance of $r_j$ away from all the primary nodes. Hence, the primary nodes is at least at the distance of $\sqrt{r_j^2 - (\frac{r_s}{2})^2}$ away from the secondary communication links. When there is an infinite vacant space outside the disks with radius $\sqrt{r_j^2 - (\frac{r_s}{2})^2}$ centered at primary nodes, these links can exist in the vacant space and they together form the percolation cluster of secondary nodes. Recall the definition of the vacant space to be infinite in the last section, the area of vacant space is of order $\Omega(1)$. Also, the space taken by each communication link is bounded by $r_s$ multiplied by a sufficiently small width, denoted as $\epsilon$ in Fig. 2. So the number of secondary communication links existing in the vacant space goes to infinity and they can form a percolated cluster over the secondary network.

Next we will prove that when the primary poisson boolean $B(n, \sqrt{r_j^2 - (\frac{r_s}{2})^2})$ is not percolated, there exists an infinite vacant space in the primary network. We can first consider this problem inversely. When the primary poisson boolean $B(n, \sqrt{r_j^2 - (\frac{r_s}{2})^2})$ is percolated, there is an infinite chain composed of the disk with radius $\sqrt{r_j^2 - (\frac{r_s}{2})^2}$ spanning over

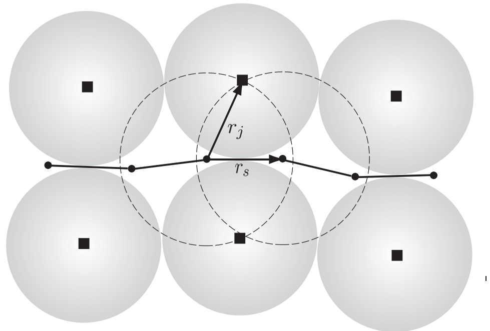


Fig. 2. Secondary users can only transmit in the vacant space. All the primary nodes are at least $r_j$ distance away from secondary nodes in the percolated cluster.

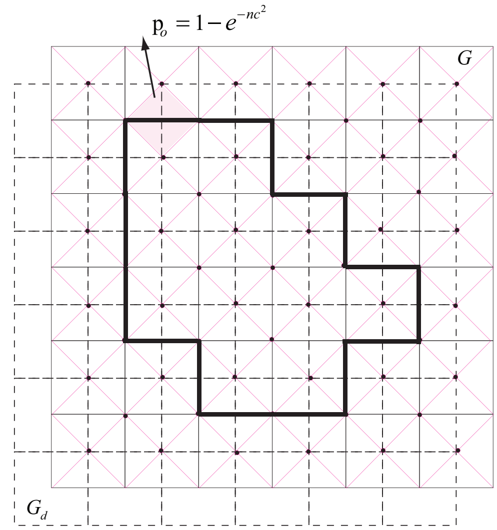


Fig. 3. Discrete grid model $G$ and dual grid $G_d$. Each edge in the grid is open with probability $p_o$. When the dual grid is not percolated, there, any given vertex in $G_d$ can be surrounded by a loop of open edges in $G$.

the whole network. Any secondary communication link cannot traverse the chain and thus all the active secondary users will probably not be connected as a whole. Adopting the standard techniques in the continuum percolation, we map the poisson boolean model into a discrete grid model to complete the proof.

To begin with the construction, we partition the network square into small squares with side length $c = \frac{1}{2}\sqrt{r_j^2 - (\frac{r_s}{2})^2}$. We define each small square to be open if there is at least one primary user inside the small square. Otherwise, the square is defined to be close. Thus, according to the poisson point process, the open probability $p_o$ for each small square is independent and it satisfies that

$$p_o = 1 - e^{\frac{1}{4}nr_s^2(\eta^2 - \frac{1}{4})}.$$

Then we add horizontal and vertical lines to connect the vertices of the small squares and they form the discrete grid $G$. For each square, two of its diagonal vertices will be connected together and we call the diagonal an open edge if the square where it is located is open. Otherwise, the edge is close. When the poisson boolean model $B(n, \sqrt{(r_j^2 - \frac{r_s}{2}^2)})$ is percolated, there exists an infinite number of connected open edges in $G$, denoted as $O_{path}$, with $|O_{path}|$ representing the corresponding number of edges contained. In contrast, when $B(n, \sqrt{r_j^2 - (\frac{r_s}{2})^2})$ is not percolated, the number of any connected open edges is finite. And in this case, we have that

$$\lim_{L\to\infty} Pr(|O_{path}| = L) = \lim_{L\to\infty} p_o^L 4 \cdot 3^{L-1} \to 0.$$

Thus we get that $p_o < \frac{1}{4}$.

For the following discussion, we construct a dual grid $G_d$ of the original one. The dual grid is formed by placing a vertex at the center of each small square in the original grid. Then we connected all the vertices in $G_d$ with horizontal and vertical lines, denoted by the dashed lines in Fig. 3. The open edge in the dual grid is defined as those which intersect with close edge in $G$. Hence, the open probability for edge in $G_d$ is $1 - p_o$. When the dual grid is not percolated, then the number of connected open edges in $G_d$ is finite and they can be surrounded by a loop of connected open edges in $G$. Let $\sigma(2L)$ denote the number of loops composed of $2L$ open edges. According to [8], $\sigma(2L)$ is bounded by $\sigma(2L) \leq (L-1) \cdot 3^{2(L-1)}$. And the probability that there exists such a loop is upper bounded by

$$Pr(\sum_{L=2}^{\infty} \sigma(2L) \geq 1) = \sum_{L=2}^{\infty} p_o^{2L}\sigma(2L) \leq \frac{9p_o^4}{[1 - 9p_o^2]^2}.$$

When $p_o < \frac{1}{4}$, the probability is smaller than 1. Thus, there exists a positive probability that the dual grid is percolated. Notice that the open edges in the dual percolation model intersect with close edges in $G$ located in small squares without primary users. Thus, the percolation cluster in the dual grid is located in the vacant space of poisson boolean model $B(n, \sqrt{r_j^2 - (\frac{r_s}{2})^2})$ and its size goes to infinity. Hence, we have proved that when $B(n, \sqrt{r_j^2 - (\frac{r_s}{2})^2})$ is not percolated, there exists an infinite vacant space outside primary rejection region. ■

Theorem 1 characterizes the restriction of the primary network topology on the secondary communication links. As communications in secondary network are limited by available OSA, the characteristics of the secondary network are, to a large extent, dependent on the value of primary node density. When the primary node density is small, there are sufficient communication opportunities for secondary users. Most secondary users are located out of rejection region, which enables them to communicate without spectrum constraints. Therefore, the impact of the primary network on the secondary communications is negligible. However, when the primary users occupy more communication bandwidth, secondary users have to be turned off due to the higher primary priority. The number of active nodes reduces when more users are located in the rejection region and thus unable to transmit. As a consequence of fewer active nodes, the connectivity of the secondary network is greatly deteriorated.

## VII. Percolation Degree

Since the pioneering work by Meester et. al. in [8], percolation theory has been demonstrated as a powerful technique in analyzing connectivity as well as other network metrics.

Different from the requirements for full connectivity that every node in the network must be connected to each other, it is only required in percolation theory that the majority of nodes in the network are connected in the same cluster, under which the network is defined as percolated. This revise in terms of connectivity loosen the constraints on network topology and therefore brings about appealing improvement on network performances. However, most previous works only focus on the phase transition phenomenon in the percolation whereas the characteristics of communication links in the percolated cluster still remains unclear. In previous sections, we have analyzed the critical primary node density to allow percolation in the secondary network. In this section, we will concentrate on the discussion of the percolation degree of the secondary network.

### A. Lower Bound

Now we consider the asymptotic $k$-connectivity in the secondary network and derive the lower bound of percolation degree. The lower bound characterizes the minimum node degree required on the secondary nodes in the percolated cluster. The strict definition can be characterized as follows.

$$l^* \triangleq \inf\{l : \exists\ n < n^* \text{ s.t. The percolation degree is } l.\}.$$

In this paper, we focus our discussion on the percolation phenomenon in the secondary network, which means that the secondary network is connected within a huge cluster. However, the difference from full connectivity is that not all the secondary nodes are connected as a whole. Instead, it is only required that the number of nodes in the huge cluster tends to infinity with the increase of $m$, which we will denote as $\zeta(m)$ satisfying $\lim_{m\to\infty}\zeta(m)\to\infty$. Considering the percolation degree of the secondary network, we aim to derive the minimum number of neighbors when there are $k$ communication links between any two nodes in the percolated cluster.

To compute the lower bound of the percolation degree, we will first give a useful theorem put forward by Pensrose in [13].

*Theorem 2 (Pensrose [13]):* In a dense network with node density of $\lambda$, let $\mu_k(\lambda)$ denote the minimum transmission radius that the network is $k$-connected. And denote by $\nu_k(\lambda)$ as the minimum transmission radius for which the node degree of the network is at least $k$. It satisfies that

$$\lim_{\lambda\to\infty}\mathbf{Pr}(\mu_k(\lambda)=\nu_k(\lambda))=1.$$

According to the Poisson Boolean Model in [4], in a randomly distributed network with poisson point process of parameter $\lambda$, the probability that there are $x$ nodes falling in a region of area $A$ is $e^{-\lambda A}\frac{(\lambda A)^x}{x!}$. Thus we can get the probability that there are $k$ nodes in the transmission range of a node located at $x$. Denoting $D(x)$ and $N(x)$ as the node degree and the number of of $x$, respectively, we have

$$\mathbf{Pr}(D(x)=k)=(\lambda\pi r_s^2)^k\cdot\frac{\exp(-\lambda\pi r_s^2)}{k!}.$$

And the mean number of points with node degree $k$ is

$$\mathbb{E}(N(x)|D(x)=k)=\lambda\int_{\mathcal{S}}\mathbf{Pr}(D(x)=k)dx.$$

Based on the result of Theorem 2, we can get the connectivity degree of the percolated cluster by finding out the minimum node degree of nodes in the cluster. Regarding the minimum node degree of the network, Pensrose proved in [13] that:

*Theorem 3 (Pensrose [13]):* Given a network with transmission range $r_s$, if $k$ satisfies the following condition

$$\lim_{\lambda\to\infty}\mathbb{E}(N(x)|D(x)=k-1)=e^{-\beta},$$

then it follows that

$$\lim_{\lambda\to\infty}\mathbf{Pr}(\nu_k(\lambda)\le r_s)=\exp(-e^{-\beta}).$$

Theorem 3 establishes a quantitative relationship between the average number of nodes with node degree $k$ and the probability of $k$-connectivity in the network. Jointly considering Theorem 2 and Theorem 3, to achieve the connectivity degree $k$ in the percolated cluster, we must first find the minimum transmission radius $r_s$ such that $\mathbb{E}(N(x)|D(x)=k-1)$ tends to zero. This result is given in the following lemma.

*Lemma 1:* In a dense cognitive network, the asymptotic transmission radius of which the secondary network to achieve $k$-connectivity in the percolated cluster is

$$r_s=\sqrt{\frac{\log\zeta(m)+k\log\log\zeta(m)}{\pi\zeta(m)}}.$$

*Proof:* From Theorem 3, the minimum transmission radius is determined by the mean number of points in the huge cluster with node degree $k-1$, which can be specified as

$$\mathbb{E}=\zeta(m)\int_{\mathcal{S}}\frac{(\zeta(m)S(x))^{k-1}}{(k-1)!}e^{-\zeta(m)S(x)}dx. \qquad (1)$$

Here $\mathcal{S}(x)$ denotes the part of the transmission disk of $x$ located inside $\mathcal{S}$. In the dense network model, there are some nodes located near the boundary of the network area, leaving the transmission disks of these nodes located partially outside $\mathcal{S}$. Hence the corresponding $\mathcal{S}(x)$ is smaller than $\pi r_s^2$. However, note that these nodes only exist in the rectangular border area with size $1-(1-2r_s)^2$, which tends to zero compared to the whole network area. In the percolated cluster, since we only consider the connected huge component composed of an infinite number of nodes, the nodes located near the boundary with finite size can be neglected. Therefore we can simplify the average number $\mathbb{E}$ as

$$\mathbb{E}=\zeta(m)\frac{(\zeta(m)S(x))^{k-1}}{(k-1)!}e^{-\zeta(m)S(x)}.$$

Substituting $r_s=\sqrt{\frac{\log\zeta(m)+k\log\log\zeta(m)}{\pi\zeta(m)}}$, we can get that $\mathbb{E}$ tends to zero under the limit of $m$. This completes our proof of Lemma 1. ■

The critical transmission radius of $k$-connectivity for secondary network percolation is smaller than that obtained in [14]. The improvement comes from the fact that the boundary effect is negligible in the percolated cluster where only the node degree of the majority of the connected components are

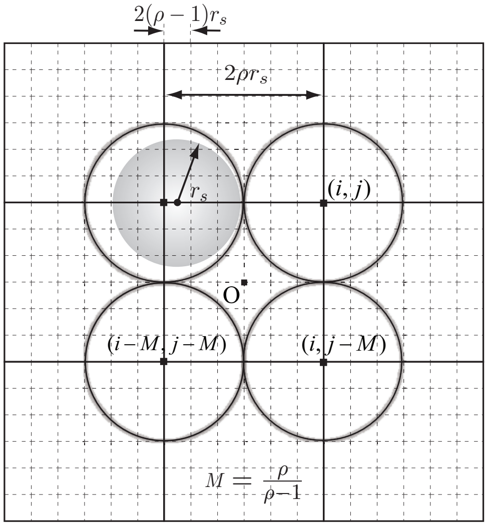


Fig. 4. Every secondary transmission disk can be covered by disk with radius $\rho$ centered at one vertex of $G'$.

considered whereas the nodes located near the border becomes a bottleneck of the $k$-connectivity in [14]. Thus, the boundary effect has little affection on the connectivity degree in the percolated cluster.

Next we will begin to derive the percolation degree of the secondary nodes in the percolated cluster. As is illustrated in Fig. 4, we divide the network square into a grid $G'$ of side-length $2(\rho-1)r_s$ with the center of the square as the origin, where $\rho$ is a constant larger than 1. Then for each secondary node in the network, it is at most at the distance of $(\rho-1)r_s$ away from one vertex of the grid. Let $M = \frac{\rho}{(\rho-1)}$, we divide the vertices of the grid into $M^2$ groups $G' = G'(i,j), i,j = 0, \cdots, M-1$, where $G'(i,j)$ is composed of nodes located at $(i+k_1M, j+k_2M), k_1, k_2 \in \mathbb{Z}$. Denote $A(G'(i,j))$ as the event that all the disks with radius $\rho r_s$ centered at vertices in $G'(i,j)$ contains no more than $l^*$. Note that since the number of groups is finite, we can deduce that when $A(G'(i,j))$ occurs with probability approaching 1, $A(G')$ is also an almost sure event.

Denote by $N(i,j)$ as the number of disks in $G'(i,j)$ and $D(i,j,n)$ $(n = 1, \cdots, N(i,j),)$ as the disk in $G'(i,j)$, with $|D(i,j,n)|$ being the number of nodes in the disk. Then the probability of $A(i,j)$ can be expressed by

$$Pr(A(G'(i,j))) = 1 - \sum_n Pr(|D(i,j,n)| > l^*). \quad (2)$$

As the number of nodes located in $D(i,j,n)$ is a random variable following poisson distribution with parameter $\lambda_D = \rho^2[\log\zeta(m) + k\log\log\zeta(m)]$, applying Chernoff bound and Stirling's Formula we can get that when $l^* = e\lambda_D$, the following probability is sufficiently small.

$$\begin{aligned} Pr(|D(i,j,n)| > l^*) &\leq \frac{e^{-\lambda_D}(e\lambda_D)^{l^*}}{(l^*)^{l^*}} \\ &= \frac{1}{(\eta(m))^{\rho^2}(\log\eta(m))^{k\rho^2}}. \end{aligned}$$

The disks in $G'(i,j)$ are at least $2\rho r_s$ distance away from each other, so $N(i,j)$ can be bounded by $N(i,j) \leq \frac{1}{\pi\rho^2 r_s^2} = \frac{\zeta(m)}{\rho^2[\log\zeta(m)+k\log\log\zeta(m)]}$. Substituting the result back into Equation (2) and employing union bound we get

$$Pr(A(i,j)) = 1 - \frac{[\log\zeta(m) + k\log\log\zeta(m)]^{-1}}{\rho^2(\zeta(m))^{\rho^2-1}(\log\zeta(m))^{k\rho^2}}.$$

When $m$ tends to infinity, the above probability tends to one. Thus, the probability that all the transmission disks in the secondary network contains no more than $l^*$ nodes tends to one. Hence if every secondary user is connected to its $l^*$ nearest neighbors, the induced secondary network topology is $k$-connected almost surely. This completes our proof of Theorem 2, stating that the lower bound of the percolation degree of secondary network to achieve $k$-percolation is $l^* = e\rho^2[\log\zeta(m) + k\log\log\zeta(m)]$.

### B. Upper Bound

In the discussion above, we have considered the minimum neighbor number for the secondary percolation cluster to achieve $k$-connectivity. When there are a large number of primary users inside the network, the transmission range of most secondary users inside the network may shrink accordingly at the border of the primary rejection disks. Thus the nodes in the secondary percolation cluster must be connected to more neighbors to ensure $k$-percolation. Considering the boundary effect of primary rejection disks, in the following we will derive the upper bound of the percolation degree.

From Theorem 1, when $n < n^*$ there is an infinite vacant component in the primary network where secondary percolated cluster can exist. When primary node density decreases, there exists more vacant space in the network area where secondary network is allowed for communication. Hence, the opportunistic spectrum available for the secondary cluster is different with regards to different primary node densities. This results in adjustment of secondary users on their transmission protocol to take better advantage of the spectrum as well as ensure the network connectivity.

We give the following theorem for the upper bound of the percolation degree.

*Theorem 4:* When $n < n^*$, let $\alpha = \arcsin\frac{1}{2\eta}$, $l^* = e\rho^2[\log\zeta(m) + k\log\log\zeta(m)]$, then the minimum neighbor number $l$ for the secondary network to ensure $k$-percolation satisfies that

1) when $\frac{p_c}{(\eta r_s)^2} < n < \frac{p_c}{r_s^2(\eta^2-1/4)}$, $l = l^*/\delta_1$, $\delta_1 \in (c_\pi, c_\pi')$, where

$$\begin{cases} \pi c_\pi = \alpha + \sin\alpha - \eta^2[2\alpha - \sin(2\alpha)], \\ \pi c_\pi' = \alpha + \sin\alpha + \eta^2[2\sin(2\alpha) + \pi - 4\alpha]; \end{cases}$$

2) when $\frac{p_c}{[(\eta+1)r_s]^2} < n < \frac{p_c}{(\eta r_s)^2}$, $l = l^*/\delta_2$, $\delta_2 \in [c_\pi', 1)$;
3) when $n < p_c/[(\eta+1)r_s]^2$, $l$ can achieve $l^*$ with a high probability.

*Proof:*

Similar to the discussion of the lower bound, as we only consider the percolation connectivity, the number of nodes located near the border of the network can be neglected since it is sufficiently small compared to the total number of nodes in the network. In the percolation phenomenon, it only needs to be ensured that a sizable number of nodes are connected as

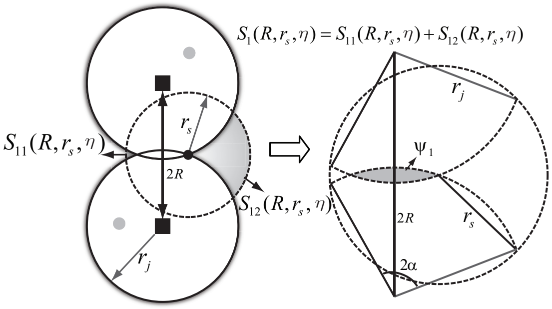


Fig. 5. The distance between two rejection disk centers is $2R$, $R = \gamma r_s$ and $\sqrt{\eta^2 - \frac{1}{4}} < \gamma < \eta$. $S_1(R, r_s, \eta)$ is composed of two parts.

a whole in a cluster. However, inside the network area where primary users exist, the boundary effect near the rejection disks cannot be neglected. This gives an explanation on why the primary network has an impact on the percolation degree of secondary network. Furthermore, by investigating this relationship we can gain a better understanding of the interaction between the two networks coexisting in the same network area. Now we will depict the cognitive network topology inside the network area by considering three situations with regard to different densities of the primary network.

First, when $\frac{p_c}{(\eta r_s)^2} < n < \frac{p_c}{r_s^2(\eta^2-1/4)}$, the network topology is illustrated in Fig. 5. There is an infinite vacant space in the Poisson Boolean Model $B(n, R)$, $r_s\sqrt{\eta^2 - \frac{1}{4}} < R < r_j$. In this scenario, an intersection area exists between two neighboring primary rejection disks in the *percolated* cluster almost surely. And the average distance $2R$ between two neighboring vertices satisfies the condition above. Denote $S_1(R, r_s, \eta)$ as the shaded area where the distance from all points inside this region to a point of intersection of two rejection disks with a distance of $2R$ ($R = \gamma r_s$, $\sqrt{\eta^2 - \frac{1}{4}} < \gamma < \eta$) is less than $r_s$. Then the intersection area between the transmission disk of a secondary node near the rejection region and vacant space is reduced to $S_1(R, r, \eta)$.

When the primary node density becomes higher, the transmission range of some secondary nodes decreases as a result of the restriction by the primary topology. To investigate this impact, we will analyze the value of $S_1(R, r_s, \eta)$ regarding the distance $R$ between two rejection disk centers. As is illustrated in Fig. 5, $S_1(R, r_s, \eta)$ is composed of two separate parts. Let $\alpha$ denote the constant vertex angle in the triangle with side $r_j$ and base $r_s$, as is noted in the right part of Fig. 5, and $\Psi_1$ the intersection area of two rejection disks. Then the area of the shaded area can be written as

$$\begin{aligned}
&S_1(R, r_s, \eta) = S_{11}(R, r_s, \eta) + S_{12}(R, r_s, \eta)\\
&= \pi r_s^2 - r_s^2(\pi - \alpha - \sin\alpha) - r_j^2[2\alpha - \sin(2\alpha)] + \Psi_1\\
&= r_s^2(\alpha + \sin\alpha) - r_j^2[2\alpha - \sin(2\alpha) - 2\theta_R + \sin(2\theta_R)]\\
&\doteq c_1 r_s^2 + 2r_j^2\theta_R - r_j^2\sin(2\theta_R).
\end{aligned}$$

Notice that $\alpha$ is a constant, so $c_1$ is also a constant which can be expressed as $c_1 = \alpha + \sin\alpha - \eta^2[2\alpha - \sin(2\alpha)]$; $\theta_R = \arccos\frac{R}{r_j} = \arccos\frac{\gamma}{\eta}$ and $0 < \theta_R < \arcsin\frac{1}{2\eta} \le \frac{\pi}{6}$. Note that $n^*(\eta^2 - \frac{1}{4})r_s^2 = nR^2$, we have $\gamma = \sqrt{\frac{n^*}{n}(\eta^2 - \frac{1}{4})}$, then $\theta_R = \arccos\sqrt{\frac{n^*}{n}(1 - \frac{1}{4\eta^2})}$. For simplicity, let $\delta_1 = \frac{S_1(R,r,\eta)}{\pi r_s^2}$ denote the ratio of the shaded area to the transmission disk. Computing the derivative of $S_1(R, r_s, \eta)$ with respect to $\theta_R$, we get

$$\frac{dS_1(R, r_s, \eta)}{d\theta_R} = r_j^2[2 - 2\cos(2\theta)] \ge 0.$$

Hence, $S_1(R, r_s, \eta)$ is a monotone increasing function with the increase of $R$. Let $c_1{}' = \alpha + \sin\alpha + \eta^2[2\sin(2\alpha) + \pi - 4\alpha$, it satisfies that $c_1/\pi < \delta_1 < c_1{}'/\pi]$.

Due to the boundary effect from rejection region, the vacant space where secondary transmitters are located is split into two parts. The border area, denoted as $SN_1$ is the space near primary rejection disks, of which secondary nodes have smaller valid transmission region. The left area, denoted as $SN_2$, is sufficiently far away from the primary nodes, making the transmission region of secondary nodes located in $SN_2$ unaffected by the primary network. As $SN_1$ is around the primary rejection disks, the area of $SN_1$ is no larger than $n\pi(r_j + r_s)r_s$. And the transmission region of secondary nodes in this area is at least larger than $S_1(R, r_s, \eta)$. For simplicity, let $\delta_1 = \frac{S_1(R,r,\eta)}{\pi r_s^2}$ denote the rate of the shaded area to the transmission disk. Rewriting Equation (1), we have

$$\begin{aligned}
&\zeta(m)\int_{\mathcal{S}} \frac{(\zeta(m)S(x))^{k-1}}{(k-1)!}e^{-\zeta(m)S(x)}dx\\
&\le \zeta(m)\frac{(\zeta(m)\pi r_s^2)^{k-1}}{(k-1)!}e^{-\zeta(m)\delta_1\pi r_s^2}\cdot n\pi r_s^2(\eta+1)\\
&\quad + \zeta(m)\frac{(\zeta(m)\pi r_s^2)^{k-1}}{(k-1)!}e^{-\zeta(m)\pi r_s^2}
\end{aligned}$$

To ensure the $k$-connectivity of the secondary percolated cluster, the above expression must tend to zero with the increase of $m$. Recall the proof of Lemma 1, we can see that when $r_s = \sqrt{\frac{\log\zeta(m) + k\log\log\zeta(m)}{\delta_1\pi\zeta(m)}}$, $\lim_{m\to\infty}\mathbb{E} \to 0$. Thus, the transmission radius of secondary users has to be increased with a factor $\frac{1}{\delta_1}$ in order to ensure $k$-percolation in a denser primary network with density of $\frac{p_c}{(\eta r_s)^2} < n < \frac{p_c}{r_s^2(\eta^2-1/4)}$.

To derive the percolation degree in this network topology, we have to take into account the impact of primary restriction on secondary neighbor degree. For every secondary user, the probability that it has $l$ neighbors located in its transmission range is also determined by the probability that all the primary users are at the distance of at least $r_j$ away from its transmission region. And the number of secondary nodes in $D(i, j, n)$ is a poisson random variable with parameter $\lambda_D{}' = \frac{\rho}{\delta_1}[\log\zeta(m) + k\log\log\zeta(m)]$. Thus, we have

$$Pr(|D(i, j, n) > l|) \le e^{-n\pi(r_s+r_j)^2}\cdot\frac{e^{-\lambda_D{}'}(e\lambda_D{}')^l}{l^l}$$

Similarly, we can get that when $l = e\lambda_D{}' = \frac{e\rho^2}{\delta_1}[\log\zeta(m) + k\log\log\zeta(m)]$, the $l$-nearest neighbor model is $k$-percolated.

When $\frac{p_c}{[(\eta+1)r_s]^2} < n < \frac{p_c}{(\eta r_s)^2}$, we can see from the scenario depicted in Fig. 4 that the distance between two rejection disks becomes larger. However there is still an intersection area between the transmission disk of an active node and the rejection disks. Similarly, we denote the shaded area as $S_2(R, r_s, \eta)$, where $\eta r_s < R < (\eta+1)r_s$. This area is the part of a transmission disk outside RR, with the center of the transmission disk being the midpoint of two rejection disk

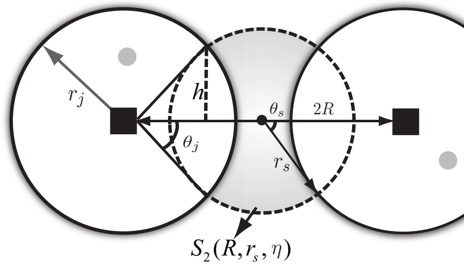


Fig. 6. The is no intersection between two rejection disks when $\eta < \gamma < \eta+1$. The shaded area denote the area of the transmission disk located outside RR.

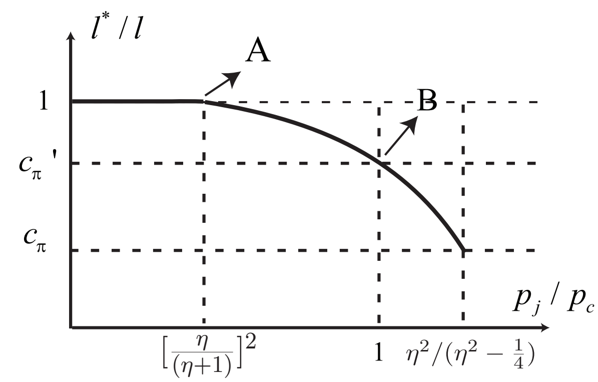


Fig. 7. The curve between the percolation degree and percolation probability $p_j$ where $p_j = nr_j^2$.

centers. For any active node, $S_2(R, r_s, \eta)$ is the lower bound of a transmission disk area outside RR.

$$\begin{aligned} S_2(R, r_s, \eta) &= \pi r_s^2 - r_j^2 \cdot 2\theta_j - r_s^2 \cdot 2\theta_s + 2Rh \\ &= \pi r_s^2 - r_j^2(2\theta_j - \sin 2\theta_j) - r_s^2(2\theta_s - \sin 2\theta_s) \\ &= \pi r_s^2 - \sum_{i=1}^{\infty} \frac{(-1)^i}{(2i+1)!}[\eta^2(2\theta_j)^{2i+1} - (2\theta_s)^{2i+1}]r_s^2 \\ &\geq r_s^2(\pi - \frac{4}{3}\eta^2\theta_j^3 - \frac{4}{3}\tan^3\theta_s) \\ &\sim r_s^2[\pi - \frac{4}{3}\frac{\eta^3}{\eta+1}\theta_j^3] \end{aligned}$$

As is denoted in Fig. 6, $\theta_j = \arccos\frac{\gamma^2+\eta^2-1}{2\gamma\eta}$, $\theta_s = \arccos\frac{\gamma^2+1-\eta^2}{2\gamma}$, and $\eta\sin\theta_j = \sin\theta_s$. The last equality holds when $\eta \gg 1$ so that $\tan\theta_s \sim \theta_s$. Notice that the last expression above is also a monotone function with respect to $\theta_j$. Thus we can get the range of $S_2(R, r_s, \eta)$.Let $\delta_2 = \frac{S_2(R,r)}{\pi r_s^2}$, then it is easy to get that $c_1'/\pi < \delta_2 < 1$. Similar to discussion of the first case, we can get that the critical secondary transmission radius to achieve $k$-percolation is $r_s = \sqrt{\frac{\log\zeta(m)+k\log\log\zeta(m)}{\delta_2\pi\zeta(m)}}$. And the percolation degree satisfies that $l = \frac{e\rho^2}{\delta_2}[\log\zeta(m) + k\log\log\zeta(m)]$.

When $n$ gets even smaller, that is $n < p_c/[(\eta+1)r_s]^2$, the whole area of a transmission disk of an active node is possibly located inside the vacant component of the primary network. Thus the percolation degree can reach the lower bound $l^*$. The result shows that there is a threshold of primary node density under which the impact of the primary network topology on the percolation of the active cluster is negligible. ■

From Theorem 4, we can see that the lower bound of the percolation degree $l^*$ of secondary network is restricted, to a larger extent, by the size of the percolated cluster and the connectivity degree in this cluster. As for the size of the percolated cluster, it is influenced by the node densities of both networks. When the primary users are sparsely distributed over the network, they incur little impact to the secondary network, where most of the secondary users can therefore transmit without restriction from the primary network. Hence, a huge percolated cluster can form and every secondary transmitter belongs to it with probability independent of each other. This corresponds to the scenario where the utilization rate of licensed spectrum is extremely low. However, when the primary node density becomes relatively higher, there will be more secondary users in the rejection region than that in the sparse primary network. To ensure $k$-connectivity in this network scenario, secondary users have to increase their own spatial density to make sure there are enough relays along the communication links. Also, when $k$ increases, more communication links are required between secondary nodes, so secondary users must be connected to more neighbors for information transmission.

For the upper bound, we consider the valid secondary transmission range influenced by the primary node density. The result varies according to different network topology. From the discussion above, it is easy to know that denser primary network has greater influence on the connectivity behavior of the secondary network. Thus, it will be more complicated for the cognitive users to adjust their transmission activities in this network scenario. However, in the cognitive network where primary spectrum utilization is extremely low, the existence of secondary users can greatly help to take better advantage of communication resources. And their communication activities are much freer compared to that in dense primary network. There exists a threshold for the primary node density below which the restrictions from the primary network to the secondary one is negligible, making the analysis of secondary network greatly simplified.

Moreover, in both cases, we consider under the condition where the primary node density is lower than the critical value proposed in Theorem 1. It is equivalent that in such circumstance the primary network is relatively sparse, being entirely possible that the distance is less than $p_c/[(\eta+1)r_s]^2$. Therefore, we consider the mostly-likely-happen case to derive the accordingly results.

## VIII. Discussion

The analysis of the percolation degree is mainly based on the primary node density and the theoretical result is illustrated in Fig. 7, showing the relationship between percolation degree and the percolation probability. In the following we give some intuitive explanations on our results. 1. *Boundary Effect Regarding the Network Square*

From the derivation of the percolation degree in Section 7, we can see that the boundary effect regarding the border of the network is quite different from that in the case of full connectivity. To achieve full connectivity, it is required that every node in the network has to be connected within at least $k$ communication links from any other nodes. As those nodes located near the network border have fewer neighbors than nodes whose transmission disk are totally located inside the network area, their node degree becomes the bottleneck of achieving the network connectivity degree. However, for the percolated cluster, we only have to make sure that the majority of the network are connected as a whole. On the other side, the number of nodes located near the network border is sufficiently small compared to the size of the whole network. Thus, the node degree of these nodes is negligible, implying that the boundary effect regarding the network border has no impact on the percolation degree.

2. *Boundary Effect Resulted from the Primary Network*

For the secondary network, the valid area of their transmission disk is also determined by the primary network topology. As primary nodes are scattered over the whole network, the boundary effect therefore spreads to the intersection of the two networks. The impact is similar to the boundary effect at the border of the network area. As this border affects more secondary users, the connectivity degree in the secondary network is greatly abated. As a result, the percolation degree becomes a function related to the primary node density.

From the curve we can see that there are two turning points concerning the relationship between the percolation degree and the primary node density. The upper bound $l^*$ is asymptotically achievable when the primary node density is lower than the critical value $\frac{p_c}{[(\eta+1)r_s]^2}$ immediately. Point $A$ shows that when there is an intersection between RR and secondary transmission disk where the percolation degree decreases. And the denser the primary network, the more quickly the percolation degree decreases. This can be explained in the way that the area of the intersection between the two networks increases more quickly in denser primary network. When the primary node density is higher than $\frac{p_c}{(\eta r_s)^2}$, as is pointed out at $B$, there is no spectrum access available between adjacent primary transmitters, leading to dramatic decrease of the percolation degree. Also the denser the primary network, the more quickly it decreases. When $n > n^*$, percolation cannot be achieved in the secondary network because there is no infinite vacant component in the primary network and the secondary clusters can only form with finite size.

Our results are applicable to the case of one-connectivity by setting $k = 1$ since the corresponding results are close to that obtained under one-connectivity in previous literatures, up to a factor of $\log\log n$. Although obtained under the two dimensional unit-area square, our results can also be employed to analyze the scenarios in higher dimensional space, such as 3D space, which is mainly addressed in [11], [15]. Many results derived in this paper can be directly generalized to 3D space. However, as neighbors of a single node may scatter in three different dimensions, we can no longer limit our analysis to the case of two objects but consider the space deployment of the nodes instead. We will leave this as our future work.

Moreover, the assumption that the primary network is static can be relaxed. In the static setting, the nodes are not allowed to use store and forward strategies, which enables the topology changes over time. As many realistic traces demonstrate that nodes are usually moving around rather than staying static, it remains interesting how connectivity performance will be like in such mobile cases. And we will also consider it as our future work.

## IX. Conclusion

We consider the cognitive radio networks where the primary and secondary networks are overlapped in a unit-area square following independent poisson point processes. Due to the heterogeneity of different network priorities of spectrum utilization as well as other communication resources, the connectivity of secondary network is restricted by those network parameters. While the percolation phenomenon exhibits a phase transition that the majority of the nodes in the network are connected immediately in a huge cluster composed of an infinite number of nodes spanning the whole network area, we further explore the nearest number of neighbors in the percolated cluster when there are $k$ vertex disjoint paths between any two of them. We analyze the primary network topology and derive the upper bound of primary node density where percolation can be achieved at the secondary network.. Then we focus on the discussion of the percolation in different cognitive network deployments. Our results give a closed-form expression of the percolation degree in the secondary network concerning various primary node density. The work provides a method to ensure the connectivity strength of percolation in the network and sheds light on the analysis of fault tolerance in cognitive networks.

## Acknowledgment

This paper is supported by National Fundamental Research Grant (No. 2011CB302701); NSF China (No. 60832005); China Ministry of Education New Century Excellent Talent (No. NCET-10-0580); China Ministry of Education Fok Ying Tung Fund (No. 122002); Qualcomm Research Grant; Shanghai Basic Research Key Project (No. 11JC1405100).

## References

[1] "Spectrum Policy Task Force Report," Federal Communications Commision, Report, Et docket No. 02-135, Nov. 2002.

[2] X.-Y. Li, P.-J. Wan, Y. Wang, C.-W. Yi, "Fault Tolerant Deployment and Topology Control Wireless Networks," in *Proc. ACM Mobihoc'03*, Maryland, USA, Jun. 2003.

[3] I. J. Mitola and J. G. Maguire,"Cognitive radio: Making software radios more personal," in *IEEE Pers. Commun.*, vol. 6, pp. 13-18, Aug. 1999.

[4] J. Berg, R. Meester, and D. G. White. "Dynamic Boolean models," in *Stochastic Process and Their Application*, vol. 69, pp. 247-257, 1997.

[5] M. Franceschetti, O. Dousse, D. Tse and P. Thiran, "Closing the Gap in the Capacity of Wireless Networks via Percolation Theory," in *IEEE Trans. Inf. Theory*, vol. 53, no. 3, pp. 1009-1018, 2007.

[6] J. F. C. Kingman, "Poisson Process," Oxford University Press, 1993.

[7] G. Grimmett, Percolation, Second ed. New York: Springer-Verlag, 1999.

[8] R. Meester and R. Roy, "Continuum Percolation," Cambridge University Press, 1996.

[9] P. Gupta and P. R. Kumar, “Critical Power for Asymptotic Connectivity in Wireless Networks,” *Stochastic Analysis, Control, Optimazation and Application, A Volume in Honor of W. H. Fleming*, pp. 547-566, 1998.

[10] W. Ren, Q. Zhao and A. Swami, “On the Connectivity and Multihop Delay of Ad Hoc Cognitive Radio Network,” in *Proc. IEEE ICC’10*, Cape Town, South Africa, May 2010.

[11] H. M. Ammari and S. K. Das, “Critical Density for Coverage and Connectivity in Three-dimensional Wireless Sensor Networks using Continuum Percolation,” in *IEEE Trans. Parallel Distrib. Syst.*, vol. 20, no. 6, pp. 872-885, 2009.

[12] C. Wang, S. Tang, X-Y. Li and C. Jiang, “Multicast Capacity of Multihop Cognitive Networks,” in *Proc. ACM MobiHoc’09*, New Orland, USA, Sept. 2009.

[13] M. D. Penrose. “On k-connectivity for a Geometric Random Graph,” Random Structure and Algorithms, vol. 15, no. 2, pp. 145-164, Sept. 1999.

[14] P. -J. Wan and C. -W. Yi, “Asymptotic Critical Transmission Radius for k-Connectivity in Wireless Ad Hoc Networks,” in *IEEE Trans. Inf. Theory*, vol. 56, no. 6, pp. 2867-2874, May 2010.

[15] X. -L. Bai, C. Zhang, D. Xuan and W. -J. Jia,“Full-Coverage and k-Connectivity (k=14,6) Three Dimensional Networks”, in *Proc. IEEE INFOCOM’09*, Shanghai, China, Jul. 2009.

[16] O. Dousse, P. Mannersalo, P. Thiran,“Latency of Wireless Sensor Networks with Uncoordinated Power Saving Mechanisms,” in *Proc. ACM MobiHoc’04*, Roppongi, Japan, May 2004.

[17] P. Gupta and P. R. Kumar, “The Capacity of Wireless Networks,” in *IEEE Trans. Inf. Theory*, vol. 46, pp. 388-404, 2000.

[18] X. -Y. Ta, G. -Q. Mao, B. -D. O. Anderson, “On the Phase Transition Width of Connectivity in Wireless Networks,” in *IEEE Trans. Mobile Computing*, vol. 8, no. 7, pp. 936-949, 2009.

[19] W. Ren, Q. Zhao, D. Cheowtirakul and A. Swami, “Connectivity of CR Networks: Proximity vs. Opportunity,” in *Proc. ACM MobiCom Workshop on CR Networks*, Sept. 2009.

[20] A. Teymorian, W. Cheng, L. Ma, X. Cheng, X. Lu and Z. Lu, ”3d Underwater Sensor Network Localization”, in *IEEE Trans. Mobile Computing*, Vol. 8, No. 12, pp. 1610-1621, 2009.

[21] B. Zhang, X. Cheng, N. Zhang, Y. Cui, Y. Li and Q Liang, ”Sparse Target Counting and Localization in Sensor Networks Based on Compressive Sensing”, in *IEEE Proc. InfoCom 2011*, pp. 2255-2263, Shanghai, China, 2011.

[22] L. Xiao, Y. Liu, and L. M. Ni, “Improving Unstructured Peer-to-Peer Systems by Adaptive Connection Establishment”, in *IEEE Trans. Comput. (TC)*, Vol. 54, No. 9, pp. 1091-1103, Sept. 2005,

[23] K. Liu, M. Li, Y. Liu, X. Li, M. Li, H. Ma, “Exploring the Hidden Connectivity in Urban Vehicular Networks”, in *IEEE ICNP 2010*, Kyoto, Japan, Oct. 2010.

**Luoyi Fu** received her B. E. degree in Electronic Engineering from Shanghai Jiao Tong University, China, in 2009. She is currently working with Prof. Xinbing Wang toward the PHD degree in Department of Electronic Engineering in Shanghai Jiao Tong University. Her research of interests are in the area of scaling laws analysis in wireless networks.

**Liang Qian** obtained his M.S. degree in Electronics and Communications from Southeast University, China, in 1998. After that, he joined Department of Electronics Engineering, Shanghai Jiaotong University. From April 2002 to October 2002, he was a visiting researcher at Institute of Information Processing, University of Kalsruhe, Germany, and received Ph.D. degree in Electrical Engineering from Shanghai Jiaotong University in 2004.

**Xiaohua Tian** received his B.E. and M.E. degrees in communication engineering from Northwestern Polytechnical University, Xian, China, in 2003 and 2006, respectively. He received the Ph.D. degree in the Department of Electrical and Computer Engineering (ECE), Illinois Institute of Technology (IIT), Chicago, in Dec. 2010. He is currently a research associate in Department of Electronic Engineering in Shanghai Jiaotong University, China. His research interests include wireless networks, application-oriented networking, Internet of Things. He serves as a Technical Program Committee member for Communications QoS, Reliability, and Modeling Symposium (CQRM) of GLOBECOM 2011 and the 6th International Conference on Wireless Algorithms, Systems, and Applications (WASA 2011).

**Huan Tang** received the B.S. degree from the Department of Electronics Engineering, Shanghai Jiaotong University, Shanghai, China, in 2010. Currently she is pursuing her Ph.D degree in University of California Davis.

**Ning Liu** received the B.S. degree from the Department of Electronics Engineering, Shanghai Jiaotongreceived the B.S. degree from the University, Shanghai, China.

**Guanglin Zhang** received the B.S. degree in Applied Mathematics from Shandong Normal University, China, in 2003, and the M.S. degree in Operational Research and Control Theory from Shanghai University, China, in 2006. Currently he is pursuing his Ph.D. degree at the Institute of Wireless Communication Technology (IWCT) in Shanghai Jiao Tong University. His research interests include capacity and connectivity of wireless networks, modeling and analysis of vehicular networks, network information theory, applied probability and stochastic process.

**Xinbing Wang** (SM'12) received the B.S. degree (with hons.) in automation from Shanghai Jiao Tong University, Shanghai, China, in 1998, the M.S. degree in computer science and technology from Tsinghua University, Beijing, China, in 2001, and the Ph.D. degree with a major in electrical and computer engineering and minor in mathematics from North Carolina State University, Raleigh, in 2006. Currently, he is a professor with the Department of Electronic Engineering, Shanghai Jiao Tong University. His research interests include resource allocation and management in mobile and wireless networks, TCP asymptotics analysis, wireless capacity, cross-layer call admission control, asymptotics analysis of hybrid systems, and congestion control over wireless ad hoc and sensor networks. Dr. Wang has been a member of the Technical Program Committees of several conferences including ACM MobiCom 2012, ACM MobiHoc 2012, IEEE INFOCOM 2009-2013.