Modeling, Analysis and Validation of Evolving Networks with Hybrid Interactions

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***Abstract*—In many real world networks, entities of different types usually form an evolving network with hybrid interactions. However, how to theoretically model such networks, along with quantitive characterizations, remains unexplored.**

Motivated by this, we develop a novel evolving model, which, as validated by our empirical results, can well capture some basic properties such as power-law degree distribution, densification, shrinking diameter and community structure embodied in most real datasets. Particularly, two types of results are presented in the paper. (i) Our proposed model, namely, Evolving K-Graph, consists of K node sets representing K different types of entities. The hybrid interactions among entities, based on whether they belong to the same type, are classified into inter-type and intra-type ones that are respectively characterized by two joint graphs evolving over time. Following our newly proposed mechanism called interactive-evolution, potential connections can be established among nodes with common features and further form a positive feedback. The superiorities of our model are three folded: good capture of realistic networks, mathematical tractability and efficient implementation. (ii) By analytical derivations, along with empirical validation on real datasets, we disclose two aspects of network properties: basic ones as power-law degree distribution, densification, shrinking diameter and community structure, as well as a distinctive one, that is, positive correlation observed in real networks, implying that a hub in one inter-type relationship network also has many neighbors in another one. An additional interesting finding is that through further comparison of models with or without interactive-evolution, the former one leads to an even earlier occurrence of network connectivity.

I. Introduction

EVOLVING networks has long been a significant research topic [2], [3], [4], [5] in discovery of typical features. Recent empirical work [6], [7] have identified some surprising properties in many real evolving networks, such as densification (the ratio of edges to nodes grows over time) and shrinking diameter (the diameter reduces over time). These properties, while reshaping the conventional view of network formation, lead to an urgent requirement on developing theoretical modeling that provides guidance to many applications like network analysis [8], [9], [10], recommendation optimization [11], [12], activity measurement [13], and resource allocation [14].

Motivated by it, some efforts have been made in proposing models that are in hope of characterizing evolving networks. However, to our best knowledge, existing models [7], [15], [23 - 25] only consider nodes of a single type (Details on these models will be discussed in Related Work), which falls short in the characterization of networks with hybrid interactions that are exhibited in real networks. A typical example belongs to academic networks, which have an underlying hybrid structure that is composed of three types of entities, i.e., authors, papers and topics. Specifically, an author working on certain research topics often have related publications (papers) of their owns, and meanwhile builds up collaborative relationship with other authors, resulting in citations among papers and interdependent cross-domain topics. Those elements, when influenced in the prescribed way with elapse of time, form hybrid interactions that can be classified into inter-type and intra-type ones. Some additional similar examples include social networks consisting of users, groups and interests, contact networks consisting of different roles of users, etc.

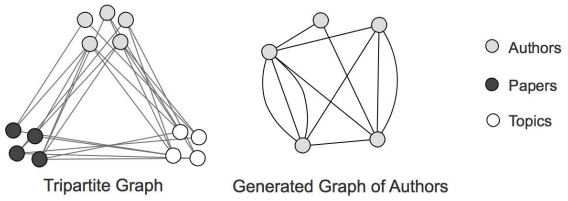

Fig. 1. An example of academic network with authors, papers and topics.

Those hybrid interactions among nodes stimulate two new features: positive correlation and interactive evolution. Some work [16], [17] revealed that the degrees of nodes in different networks are often positively-correlated. In other words, a hub in one network also has many neighbors in another one [18]. In addition to static interactions described above, there also exist dynamic ones which result in a distinctive network evolution, where the arrival of the node of a certain type may lead to the evolution of the whole network. All these properties give rise to the fact that existing models are inadequate to characterize such kind of networks. And motivated by this, we aim to model the evolving networks with hybrid interactions and disclose the theoretical properties of them. The main challenge of the work is brought by the hybrid interactions among nodes of different types. Multiple types of connections diversify the construction of network and more importantly, they are interdependent and further related to another dimension, i.e., time, which causes great difficulty in modeling and the corresponding analysis.

To address the aforementioned problems, we propose a novel model, namely, Evolving K-Graph, which uses two jointly evolving graphs, i.e., K-partite graph and generated graph, to characterize inter-type and intra-type interactions among nodes respectively, and establishes the evolving process

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of them. The network structure is illustrated for an academic network in Fig. 1, where inter-type connections among authors, papers and topics are characterized by a tripartite graph as shown in the left and the generated graph characterizing connections among authors is shown in the right. The evolving process of the proposed model incorporates two steps: self-evolution characterizing the evolution of the node set itself and interactive-evolution modeling the influence of it on other ones. In the former step, the newly arrived node first attaches a community and then picks a node from it following the preferential attachment [4], [15], which models user behaviors and leads to a multiplicative process giving community structure and power-law degree distributions. In addition to network expansion, it also allows edge leaving and node leaving. As for interactive-evolution, it follows a folklore wisdom that the correlation is transitive, that is, two nodes sharing a common neighbor are much likely to be correlated as well, and thus the arrival of a new node can further influence its neighbors of other types and even connections between them.

The superiorities of Evolving K-Graph lie in three aspects: Firstly, it can well capture realistic networks with properties such as power-law degree distribution, densification, shrinking diameter, evolving community structure, as well as positive correlation observed among different types of nodes. Secondly, the model is a mathematically tractable one that is amenable to conduct theoretical analysis. And thirdly, it is efficiently implementable, since the computational complexity of the model is of constant order in each time slot.

We further analyze the network properties of Evolving K-Graph. Our results show that the node degree is power-law distributed and grows over time with a polynomial rate. Besides, we also observe phenomenons that are prevalent in many large-scale real datasets, i.e., network density increases and network diameter shrinks or stabilizes over time. Moreover, the network is structured with communities, which remain stable over time unless some events, such as merge and split, happen. Last but not least, in Evolving K-Graph, nodes of the same type exhibit positive correlation, where both the node degrees in K-partite graph and that in generated graph are positively-correlated and the distributions of them show the same feature. This property further provides an improved performance of network connectivity: the phase transition time, i.e., the threshold time when the giant component emerges, of our model is reduced by a factor of c-power, where $c \leq 1$ is a scalar constant.

Our main contributions are summarized as follows:

- **Modeling:** We propose a novel model, Evolving K-Graph, that can well capture the properties widely observed in real evolving networks: power-law degree distribution, community structure, densification, shrinking diameter, and positive correlation. The model has significant superiority in analysis with mathematical tractability and efficient implementation.
- **Analysis:** We theoretically analyze the properties of Evolving K-Graph and results show that it speeds up the evolution in terms of growth of node degree, densification and shrinking diameter. In addition, our model reproduces community structure that remains stable over time in general but evolves if merge or split events happen. Last but not least, the model establishes positive correlation, based on which the phase transition time is reduced by a constant power.
- **Validation:** We empirically observe the properties of evolving networks with hybrid interactions on two real datasets. Then, we conduct validation on the proposed model to show that it can well reproduce the corresponding properties, from which we demonstrate that our model can well capture the features of real networks.

The rest of this paper is organized as follows. We give literature review in Section II. We present feature observations in Section III. The proposed model is described in Section IV and the network properties are analyzed in Section V. We make an extension of our model in Section V-D. Our model is validated in Section VI. We conclude in Section VII.

II. Related Work

1) Properties and models in evolving networks with uniform interactions: To begin with, we briefly introduce the properties of evolving networks with uniform interactions, which include: *Power-law degree distribution:* node degree follows power-law distribution, i.e., the probability that a node's degree d equals to k is $\mathbb{P}\{d = k\} \propto d^{-\alpha}$, which has been found in the Internet [19], the web [20] and etc; *Community Structure:* nodes in a same community are more closely connected than those cross different communities [21]; *Densification:* the ratio of number of edges to that of nodes grows over time [6], [7]; *Shrinking Diameter:* the network diameter reduces as time goes by [7]. Many efforts have been made to capture the above properties. The majority of existing works takes advantage of *preferential attachment* [22] that follows the rule: at each time slot a new node arrives the network and connects to an existing node with the probability proportional to its degree, which gives *power-law degree distribution* but fails to reproduce the other properties mentioned above. Based on this model, some variations, all somehow present a "richer gets richer" style, are proposed. In particular, some models can further cover *densification* and *shrinking diameter* properties. Kumar *et.al* propose "copying model" [23], behind which basic idea is that a new web page is often made by copying an old one, and then changing some of the links, or in other words, at each time slot, a vertex u is added and given d connections by picking a "prototype" and copying its edges. Leskovec *et.al* use a matrix operation, i.e., Kronecker product, to generate evolving graph. The proposed model [24], "Kronecker Graph", defines the Kronecker product of two graphs as that of their adjacency matrices, through which to create self-similar graphs recursively. Lattanzi *et al.* [15] propose "Affiliation Networks" model that consists of two types of nodes, i.e., actor and attribute, where the latter one is a feature capture of the former one, and based on the correlation between them the evolving process of the model is developed. In addition, *Community Structure* is reproduced in some other models. For example, Leskovec *et.al* present "Forest Fire Model" [7], where a new node links to a node w uniformly at random, picks some neighbors of w that were not visited yet and applies this step recursively. As a result, the model obtains a structure of communities-within-communities as a tree. However, all the above models assume nodes are of

TABLE I
COMPARISON OF DIFFERENT MODELS OF EVOLVING NETWORKS

Network Properties	#1	#2	#3	#4	#5	#6	Parameter Space
Preferential Attachment	No	No	Yes	No	No	No	$(0,1) \times (0,\infty)$
Copying Model	No	No	Yes	Yes	Yes	No	$(0,\infty)^2 \times (1,\infty) \times (0,1)$
Kronecker Graph	No	No	Yes	Yes	Yes	No	$[0,1]^2$
Forest Fire Model	No	No	Yes	Yes	Yes	Yes	$[0,1]^2$
Affiliation Network	No	No	Yes	Yes	Yes	No	$[0,1]^2 \times [0,\infty)^2$
Multi-typed Affiliation Network	Yes	No	Yes	Yes	Yes	No	$[0,1]^K \times [0,\infty)^{K!}$
Evolving K-Graph	Yes	Yes	Yes	Yes	Yes	Yes	$\left[\frac{2}{3},1\right] \times [0,\infty)^{K!} \times [0,1]^{K+M}$

[1] #1 - #6 are labels of properties, where #1 is "multi-typed nodes", #2 is "interactive evolution", #3 is "power-law degree distribution", #4 is "densification", #5 is "shrinking diameter" and #6 is "community structure".
[2] Parameter K denotes the number of node types and parameter M denotes the number of communities in the network.

single type, which cannot fully characterize the real networks with hybrid interactions.

2) Models of evolving networks with hybrid interactions: To our best knowledge, there are no existing works that attempt to model evolving networks with hybrid interactions. So, could models of single-typed evolving networks still well regenerate multi-typed evolving ones without an explicitly hybrid model? For example, "Affiliation Networks" model can be modified to include multi-typed nodes by adding multiple attributes/actors, where we name the modified model as "Multi-typed Affiliation Networks" and it will be used in the later comparison with our model. However, though mathematically tractable and efficient implementable as ours, such a modification ignores interactive evolution. As illustrated in academic networks that has three types, i.e., authors, papers and topics, "Multi-typed Affiliation Networks" assumes a new arrived node, e.g., paper, connects to some existing authors and topics, which, however, ignores possible links between the picked authors and topics who are also potentially related since they share a common paper. Such generation of potential links are called *interactive evolution*. Different from such a simple modification, our proposed model introduces multi-typed nodes, as well as interactive evolution among them, which leads to some distinctive features such as positive correlation and improvements of network connectivity, which will be illustrated in details in the remaining parts. And for the convenience, we present Table I to show the observed properties and whether the discussed models support them.

III. EXPERIMENTAL OBSERVATIONS

Multiple types of nodes and hybrid interactions among them may bring some new features to network properties, and in this section we try to characterize them based on two real datasets.

A. Dataset Description

Statistical properties of the datasets are summarized in Table II, where $B(X,Y)$ denotes the bipartite graph of types X and Y. In addition, some basic descriptions are presented below.

1) Academic Network: This dataset is collected from Microsoft Academic Graph in Data Mining field [25]. The dataset includes three types of nodes, i.e., authors, papers and topics, and interactions among them. In addition, every node in the dataset has a timestamp recording the time it joins the network, ranging from the year 1816 to 2015.

2) Contact Network: This real dataset is obtained from SocioPatterns [26], collected in 2013. It gives temporal contacts of the students from three classes in a high school in France. The time duration of this dataset is 5 days, where 20-second interval is regarded as a timeslot and thus the total number of timeslots is $360,700$. In this dataset, students of different classes are viewed as different types and once a contact occurs between two students we regard them as connected.

B. Settings

In both Academic Network and Contact Network, each item is formed as $< u, v, t >$, which means an edge between node u and node v is created at time t. When a node v_0 appears in an item whose timestamp is t_0, we regard this as "node v_0 is active at time t_0". Denote the smallest and largest active time of node v as t_{small} and t_{large}, the arrival time and leaving time of v are defined as follows:

- *Arrival time:* t_{small}; - *Leaving time:* $t_{large} + t_{wait}$,

where t_{wait} is a time interval that is set to ensure the node is not active anymore. In particular, t_{wait} is set as 5 and $30,000$ for Academic Network and Contact Network, respectively.

C. Observations

We first verify that some properties, which exist in ordinary evolving networks, are also observed in the multi-typed ones. Then, we introduce a distinct one, i.e., positive correlation.

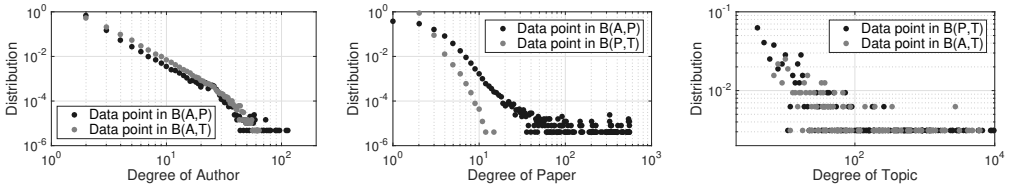
1) Basic features: Firstly, let us come to the discussion on power-law degree distribution. To study the cross-type edges, we explore the bipartite graphs that are used to model the links between nodes of different types. For example, in academic network, we use a bipartite graph $B(A,P)$ to characterize the edges between authors (A) and papers (P), where the degree of an author in A represents the number of papers he published and the degree of a paper in P indicates the number of authors of the paper. We measure the degree distributions of the two types of nodes in all bipartite graphs of academic network and the corresponding results are given in Fig. 2, from which we can observe that all the degrees are power-law distributed.

In addition, densification and shrinking diameter are another two common features that reflect how networks evolve. Results in both two datasets show that generated networks (networks that consist of nodes of the same type and intra-type connections among them) densify over time as shown in Fig. 3 and the diameters of them shrink or stabilize over time as given in Fig. 4. This observation exactly meets the general view.

2) Community structure: We conduct experiments on the dataset of academic network and use Louvain algorithm [27] to make the community partition. Results show that community structure exists and further, it evolves over time. For example, communities 1 and 2 merge into community 3 at time $t = 2002$ as shown in Fig. 5 (a) and in Fig. 5 (b), community 4 splits into communities 5 and 6 at time $t = 2012$.

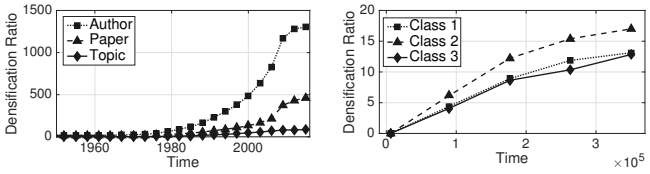
TABLE II
STATISTICAL PROPERTIES OF DATASETS.

	Properties of node sets				# of Edges			# of Timeslots
Academic Network	Physical meaning	Author	Paper	Topic	$B(A,P)$	$B(P,T)$	$B(A,T)$	200
	Notation	A	P	T	3,291,293	3,921,512	1,251,915	
	# of Nodes	1,048,576	1,042,280	403				
Contact Network	Physical meaning	Class 1	Class 2	Class 3	$B(C1,C2)$	$B(C1,C3)$	$B(C2,C3)$	360,700
	Notation	$C1$	$C2$	$C3$	1,180	946	1,833	
	# of Nodes	29	38	32				



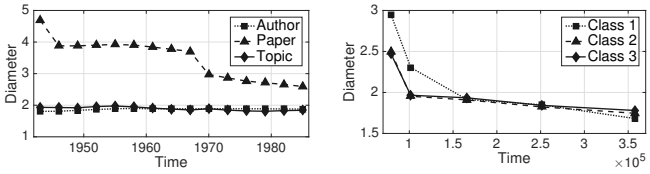
(a) Degree distribution of author (A). (b) Degree distribution of paper (P). (c) Degree distribution of topic (T).

Fig. 2. Degree distributions in dataset of academic network that consists of three types of nodes, i.e., authors (A), papers (P) and topics (T).



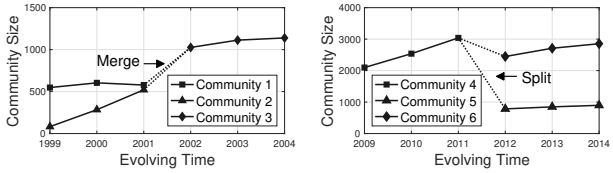
(a) Academic network. (b) Contact network.

Fig. 3. Densification property observed in the two datasets.



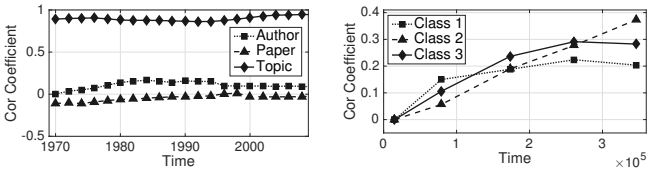
(a) Academic network. (b) Contact network.

Fig. 4. Shrinking diameter property observed in the two datasets.



(a) Merge event. (b) Split event.

Fig. 5. Merging and splitting events in community evolution.



(a) Academic network. (b) Contact network.

Fig. 6. Correlation coefficients observed in the two datasets.

3) Positive correlation: In addition to the above conventional features, we further observe a distinctive one - a hub in one bipartite network also has many neighbors in another one. We use a metric, i.e., correlation coefficient, to measure it. In the example of academic network, a positive correlation coefficient of authors indicates that an author with more papers is likely to works on more topics. We only give the intuitive explanation of the metric here. The mathematic definition of it is provided in Definition 5, Section V-C. Results of Academic Network are shown in Fig. 6 (a), where we can observe that the correlation coefficient of topics is obviously positive which even approaches 1 after the year 2010. The cor coefficient of authors is also a positive one, although not particularly large. Someone may wonder that why the correlation coefficient of papers floats around 0. This lies in the fact that the number of authors and that of topics in a paper are always limited by the publisher, ranging from 3 to 5 approximately. As for the contact network, cor coefficients of Class 1, Class 2 and Class 3 are positive and all of them approach more than 0.2 at time $t = 2 \times 10^5$ as shown in Fig. 6 (b). Based on the above results, we conclude that the node degrees are positively-correlated in networks with hybrid interactions.

IV. EVOLVING K-GRAPH MODEL

Based on the above observations, we develop a novel model named Evolving K-Graph. In the model we assume that every node is born with two attributes:

– *Type:* a set of nodes with certain functions and structures, where the k-th type is denoted by V_k, $1 \leq k \leq K$.
– *Community:* a group of nodes of the same type that have a larger probability to connect to each other than the rest, where the m-th community is denoted by C_m, $1 \leq m \leq M$.

Particularly, in our model we consider the network consisting of K types, where $K \geq 3$, and M communities. We remark that there could be multiple ways of community partition and here we assume that all types of nodes are partitioned into communities according to a same criterion. Consequently, as we will show in later description of our modeling in Section IV-B, a node of one specific type can always find, in the same community, its counterpart of a different type.

To better distinguish the conception of type and community aforementioned, let us, for example, consider academic net-

works. The networks are composed of three types, i.e., author, paper and topic, and meanwhile partitioned into communities according to their research domains, such as Mathematics, Biology and History. Note that whichever type a node belongs to, it can be partitioned into one of the provided communities according to its research domain. And obviously, the majority of the citations are more likely to be formed among papers that belong to the same domain than those across domains, and this leads to the formation of a community in citation networks. The similar cases also hold for authors in co-author networks and topics in inter-disciplinary networks.

A. K-Graph Model

The intuition behind the model is that, entities belonging to the same type are usually connected together if they are related to a common entity of other types. For examples, in academic networks, authors collaborating a same paper are often friends. We use two graphs, i.e., K-partite graph and generated graph, to characterize inter-type and intra-type relationships among nodes respectively, and model them as follows:

1) **K-partite Graph $T(V_1, ..., V_K)$:** The relationship between any two different node sets V_i and V_j can be naturally viewed as a bipartite graph $B(V_i, V_j)$ and the union set of all the bipartite graphs forms a K-partite graph $T(V_1, ..., V_K)$, which we use to characterize the inter-type connections among different types of nodes. The degree of node $v \in V_k$ is represented by a vector whose i-th component, d_{ki}, is the degree of v in $B(V_k, V_i)$.
2) **Generated Graph $G(V_i|V_j)$:** The graph $G(V_i|V_j)$ characterizes the intra-type connections among nodes in V_i, which is generated from $B(V_i, V_j)$ following the rule that *an edge exists between $v_1, v_2 \in V_i$ in $G(V_i|V_j)$ if and only if they share a common neighbor $u \in V_j$ in $B(V_i, V_j)$*. Note that in $G(V_i|V_j)$, multiple edges may exist between $v_1, v_2 \in V_i$ if they share more than one neighbor in $B(V_i, V_j)$.

This model has its applicability to massive real scenarios. Take academic networks again for instance. Denoting authors, papers and topics by V_1, V_2 and V_3 respectively, the model can then be interpreted as follows. For the tripartite graph $T(V_1, V_2, V_3)$, an edge exists between $v_1 \in V_1$ and $v_2 \in V_2$ indicates that the author v_1 published the paper v_2, and the other two kinds of edges have a similar physical meaning. For the generated graph $G(V_1|V_2)$, two authors have a common neighbor in $B(V_1, V_2)$ indicates that they collaborate a paper and thus are correlated. In addition, this model can also be used to characterize some other realistic networks such as social networks that consist of users, groups and interests, contact networks that consist of different roles of users, etc.

B. Evolving Process

In our model, K-partite graph and generated graph follow a joint evolution. We give an illustration on the evolving process of the former one, based on which that of the latter one can be obtained, and therefore we omit it here for concision. The parameters that will be used later are defined as follows:

– *Community Probability p:* the probability that a newly arrived node attaches to the community it belongs to.
– *Community Control Set $\{\beta_1, \beta_2, ..., \beta_M\}$:* a set of probabilities where $\beta_m \in (0, 1)$ denotes the fraction of nodes belonging to community C_m.
– *Arriving Probability α_k:* the probability that a new node of type V_k arrives at each time slot, where $\alpha_k \in (0, 1)$.

The effects of these parameters on the network structures are illustrated as follows. A larger *Community Probability* p gives a more evident community structure, where, specifically, when $p = 1$ there are no cross-community edges. *Community Control Set* $\{\beta_1, \beta_2, ..., \beta_M\}$ determines the scale of communities, and a larger β_m results a larger size of community C_m. In a similar way, *Arriving Probability* α_k determines the scale of nodes of type k. With these parameters, the evolving process of $T(V_1, ..., V_K)$ is presented in Algorithm 1. At time $t = 0$, an initial graph $T_0(V_1, ..., V_K)$ is given. Then at each time slot $t > 0$, a new node arrives at V_k with probability α_k and results in a two steps evolution:

Self-evolution: The new node first conducts the community attachment following the rule that chooses its own community with a larger probability p and others with a smaller one $1 - p$. Then, following the preferential attachment [15], the new node picks an existing node from the chosen community as prototype with probability proportional to its degree and copies a constant number, i.e., c_{ij} where $i \neq j \in \{1, ..., K\}$, of edges from the picked prototype. In addition, $\hat{c}_{ki}$ edges, randomly and uniformly picked in $B(V_k, V_i)$, $k \neq i$, leave the network.

Interactive-evolution: Additional edges are created following the rule that: *For each newly added node in $T(V_1, ..., V_k)$, all its neighbors in different node sets connect to each other.*

We note that the interactive-evolution is a distinctive process and a unique phenomenon that is only considered in Evolving K-Graph. It differs from the self-evolution whose evolution of a certain node set only results in the creation of edges between itself and other types. Instead, some edges among other types, not directly connected to the node type itself, will be created. Consider again the example of academic networks. The arrival of a new author merely establishes edges between it and some papers/topics in models without interactive-evolution. While in our Evolving K-Graph, it further generates edges between the papers and topics that simultaneously related to a same author.

Fig. 7 shows an example of the evolving process of tripartite graph $T(V_1, V_2, V_3)$ when a node $v \in C_1$ arrives in V_1. Note that in order to present all edges in a plane, we unfold the tripartite graph where the two node sets V_3 are actually the same one. The evolving process of $T(V_1, V_2, V_3)$ happens in two steps. In the first step, the node v chooses its own community, i.e., C_1, picks two existing nodes as prototypes and copies $c_{13} = 1$ and $c_{12} = 2$ edges from them respectively. Then, in the second step, neighbors of the new node v in V_2 and V_3 are connected together and thus two additional edges are added to the graph. And finally, one node and five edges are added and $T(V_1, V_2, V_3)$ evolves from the left-most one in the figure to the right-most one during this time slot.

For convenience, we present Table III to list all notations that will be used in later analysis, proofs and discussions.

Algorithm 1 Evolving process of $T(V_1, ..., V_K)$

Fix parameters including community probability $p \geq \frac{2}{3}$, community control set $\{\beta_1, \beta_2, ..., \beta_M\}$ and arriving probability α_k, $k \in \{1, ..., K\}$.
Fix parameters $c_{ij}, \hat{c}_{ij} \geq 0$, where $i \neq j \in \{1, ..., K\}$.

At time $t = 0$:
The initial K-partite graph $T_0(V_1, ..., V_K)$ is given, where each node in V_k has at least c_{ki} edges in bipartite graph $B(V_k, V_i)$.

At time $t > 0$:
(Evolution of V_k)
Arrival: A new node v of community C_m arrives with probability $\alpha_k \beta_m$ and is added to the node set V_k, $k \in \{1, ..., K\}$.
Self-evolution in $B(V_k, V_i)$***:***
(Community Attachment) For the bipartite graph $B(V_k, V_i)$, $k \neq i$, the node v chooses a community following the rule that choosing its own community C_m with probability p and others with probability $1 - p$, where community C_i, $i \neq m$, is chosen with the probability proportional to the parameter β_i.
(Preferential Attachment) For every node set V_i, $i \neq k$, a node $u \in V_k$ of the chosen community is picked as the prototype for the new node v with a probability proportional to its degree in $B(V_k, V_i)$. Then, c_{ki} edges are copied from u, that is, c_{ki} neighbors of u denoted by $v_1, ..., v_{c_{ki}}$, are picked uniformly and randomly and the edges $(v, v_1), ..., (v, v_{c_{ki}})$ are added to the graph.
Interactive-evolution in $B(V_i, V_j)$***:*** $c_{ki}c_{kj}$ edges are added between nodes $v_1, ..., v_{c_{ki}} \in V_i$ and $v_1, ..., v_{c_{kj}} \in V_j$, where $i \neq j \neq k$.
Edge Leaving and Node Leaving: $\hat{c}_{ki}$ edges, which are randomly and uniformly picked in $B(V_k, V_i)$, $k \neq i$, leave the network. A node leaves the network if all its edges left the network.

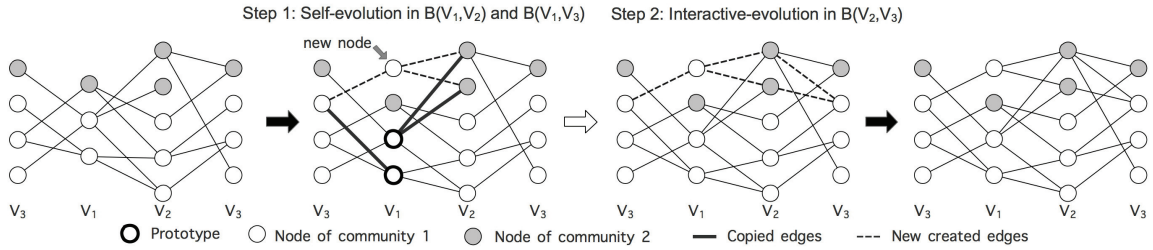


Fig. 7. An example of the evolving process of tripartite graph $T(V_1, V_2, V_3)$.

TABLE III
NOTATIONS AND DEFINITIONS

Notation	Definition
K	Total number of node types in the network.
M	Total number of communities in the network.
n	Number of nodes in the network.
$\boldsymbol{d}(t)$	Node degree at time t.
V_i	Node type i.
C_m	Community m.
α_i	Probability that a new node arrives at V_i.
β_m	Fraction of nodes belonging to C_m.
γ_i	Parameter of growth rate of node $v \in V_i$.
c_{ij}	Number of edges copied from prototype in $B(V_i, V_j)$.
$B(V_i, V_j)$	Bipartite graph with node sets V_i and V_j.
$G(V_i \vert V_j)$	Generated graph of V_i obtained from $B(V_i, V_j)$.
$\boldsymbol{G}(V_i)$	Generated network of V_i.
$T(V_1, ..., V_K)$	K-partite graph with node sets V_1, ..., V_K.

C. Complexity

We now analyze the computational complexity of Evolving K-Graph. At each time slot, the evolving process of a certain node set V_k in $T(V_1, ..., V_K)$ includes two steps. The computational complexity of each of them is analyzed as follows:

1) Self-evolution: The evolving process in this step can be divided into three parts: community attachment, preferential attachment and edge/node leaving. In the first part, a newly arrived node chooses a community following the rule that choosing its own community with a constant probability (Case 1) and others with probabilities proportional to the parameter β_i (Case 2). Obviously, the computational complexity of this operation can be achieved as $\Theta(1)$ if Case 1 happens, since it is equal to tossing a biased coin; and similarly in Case 2, this operation can also be conducted with $\Theta(1)$ complexity by using a "Probability Pool", where each community occupies a portion of the whole pool whose area is proportional to its size. Then, the choice can be achieved by randomly dropping a seed to the pool and picking the community whose area the seed falls into. Specifically, when the new node $v \in C_m$ makes the choice, one can easily realize "Probability Pool" by a linked list. Firstly, build a linked list whose length is set as the total size of communities and community C_i occupies $\frac{\beta_i}{1-\beta_m}$ fraction of it. Then, randomly and uniformly choose an item from the linked list and pick the corresponding community. Following this method, we can achieve community attachment with only $\Theta(1)$ complexity. Besides, the complexity of "Probability Pool" maintenance is also $\Theta(1)$. For the linked list realization method, at each time slot it only needs to insert constant number of new items and thus the cost is $\Theta(1)$. In the second part, i.e., preferential attachment, the new node v in V_k picks an existing node of the chosen community as prototype with probability proportional to its degree, which can also be achieved with $\Theta(1)$ complexity by using the "Probability Pool". In addition, the computational complexity of edge copying is also $\Theta(1)$ since it only involves c_{ki} times edge copying. And finally, edge/node leaving costs $\Theta(1)$ complexity since a constant number, i.e., $\hat{c}_{ij}$, of edges are removed. Combining the above three parts we have proved that computational complexity of self-evolution is $\Theta(1)$.

2) Interactive-evolution: In this step there are $c_{ki}c_{kj}$ times occurrence of edge copying, and the corresponding computational complexity is $\Theta(1)$.

Combining the above two steps, we conclude that *the*

computational complexity of our model at each time slot is $\Theta(1)$. We note that the computational complexity of our model is a linear one, which is superior to some other existing models. For example, in a milestone work [7] on network evolution modeling, Leskovec *et al.* propose the forest fire model where at each time slot the new node forms links based on a tree structure with $\Theta(\log n)$ complexity. To summarize, our model is an efficiently implementable one.

V. Network Properties

In this section we provide corresponding property analysis. We start with the case $K = 3$, and then extend our results to a more general case where $K \geq 3$ in Section V-D. In addition, though our model can cover a general evolving process, to enable mathematical tractability, the analysis below considers a network expansion mode where in each time slot, the number of newly generated edges is larger than that of left ones.

A. Properties on Evolution

This subsection presents theoretical analysis on the model's capability to capture some basic network features, i.e., power-law degree distribution, densification and shrinking diameter. For each feature, we discuss it follow a pattern of motivation, theorem, proof and implications.

1) ***Result 1: Growth of node degree****:*

We first present a theorem on growth of node degree, which plays a fundamental role in the following analysis.

Theorem 1: In the tripartite graph $T(V_1, V_2, V_3)$ generated after t time slots, for the node v of community C_m, the degree of $v \in V_1$ at time $t \geq t_0$ satisfies

$$\boldsymbol{d}(t) = \boldsymbol{d}(t_0)\left(\frac{t}{t_0}\right)^{\gamma_{1m}},$$

with the initial condition that node v is added to the set V_1 at time t_0 with degree $\boldsymbol{d}(t_0)$. This result also holds for node $v \in V_2$ and $v \in V_3$ with a symmetrical expression.

Proof: Before the calculation of node degree, let us first consider the average number of edges between nodes in V_1 and V_2 of community C_m, denoted by $e_{m12}(t)$. At each time slot, $e_{m12}(t)$ may increase in the following three cases:

Case 1: A new node arrives at V_1 with probability α_1. If the node belongs to community C_m (this event happens with probability β_m), each of its c_{12} edges connects to node of community C_m with probability p and thus in this case, $\alpha_1\beta_m c_{12}p$ edges are added; otherwise, the node belongs to the other communities. For example, assume the node is of community C_i (this event happens with probability β_i), then it has $(1-p)c_{12}$ edges connecting to nodes in other communities and $\frac{\beta_m}{1-\beta_i}$ of them connecting to nodes of community C_m, and thus in this case, $\alpha_1\beta_i c_{12}(1-p)\frac{\beta_m}{1-\beta_i}$ edges are added. Sum over all possible i and we have that in this case

$$e_{m12}(t) = C_{12}^1 t,$$

where $C_{12}^1 = \alpha_1\beta_m c_{12}\left(p + (1-p)\sum_{i\neq m}\frac{\beta_i}{1-\beta_i}\right)$. We use C_{12}^1 to denote the coefficient of $e_{m12}(t)$, with subscript indicating the edges between node sets V_1 and V_2 and superscript indicating the trigger event that the new node arrives at V_1. Note that C_{12}^1 is a constant only related to parameter m.

Case 2: A new node arrives at V_2 with probability α_2. Following the same method as in the previous case, we can obtain that in this case

$$e_{m12}(t) = C_{12}^2 t,$$

where $C_{12}^2 = \alpha_2\beta_m c_{21}\left(p + (1-p)\sum_{i\neq m}\frac{\beta_i}{1-\beta_i}\right)$ is also a constant only related to parameter m.

Case 3: A new node arrives at V_3 with probability α_3. In this case, the increase of $e_{m12}(t)$ results from the interactive-evolution. If the new node belongs to community C_m, it has $c_{31}p$ edges connecting to nodes of community C_m in V_1 and $c_{32}p$ edges connecting to that in V_2 on average. Then, these nodes are also connected due to the interactive-evolution which results in $c_{31}c_{32}p^2$ newly added edges. In the other case where the new node belongs to the other communities, for example, community C_i, it has $\beta_i c_{31}(1-p)\frac{\beta_m}{1-\beta_i}$ edges connecting to nodes of community C_m in V_1 and $\beta_i c_{32}(1-p)\frac{\beta_m}{1-\beta_i}$ edges connecting to that in V_2 on average. Take summation over all possible is and we have that

$$e_{m12}(t) = C_{12}^3 t,$$

where $C_{12}^3 = \alpha_3 c_{31}c_{32}\left(\beta_m p^2 + {\beta_m}^2(1-p)^2\sum_{i\neq m}\frac{\beta_i}{1-\beta_i}\right)$ with the similar meaning to others stated earlier.

In addition to increase, the value of $e_{12}(t)$ may also decrease in the following case:

Case 4: Edge leaving of edges in $B(V_1, V_2)$ of community C_m. According to the evolving process, where in each time slot $\hat{c}_{12}$ edges leave the network, we have that the number of edges decreased due to edge leaving in all t time slots is

$$e_{m12}(t) = -\hat{c}_{12}t.$$

Combining all the above four cases, we have

$$e_{m12}(t) = \left(C_{12}^1 + C_{12}^2 + C_{12}^3 - \hat{c}_{12}\right)t.$$

Note that the average number of edges between nodes in V_1 and V_2 grows linearly with evolving time t and its coefficient is a constant that is only related to parameter m.

Based on the above results, now we come to the calculation of node degree in tripartite graph $T(V_1, V_2, V_3)$. At each time slot t, the degree of node $v \in V_1$ in $B(V_1, V_2)$, i.e., $d_{12}(t)$, may increase in the following two cases:

– A new node arrives at V_2 and is connected to v, which results in $d_{12}(t) = d_{12}(t-1) + 1$.
– A new node arrives at V_3 and is connected to v, then v will connect to c_{32} neighbors of the new node in V_2 which results in $d_{12}(t) = d_{12}(t-1) + c_{32}$.

In preferential attachment, the endpoint of any edge is chosen with equal probability as a destination of the new edge, and thus the probability that a newly added edge in $B(V_1, V_2)$ points to node v is $\frac{d_{12}(t-1)}{e_{12}(t-1)}$, where $e_{12}(t-1)$ denotes the average number of edges in $B(V_1, V_2)$ at time $t-1$.

In addition, $d_{12}(t)$ may decrease in the following case:

– One of its edges leaves, with the probability $\frac{\hat{c}_{12}}{e_{12}(t)}$, which results in $d_{12}(t) = d_{12}(t-1) - 1$.

Combining the above cases, we have

$$d_{12}(t)-d_{12}(t-1)=\left(C_{12}^{2}-\hat{c}_{12}\right)\frac{d_{12}(t-1)}{e_{12}(t-1)}+C_{12}^{3}\frac{d_{13}(t-1)}{e_{13}(t-1)}, \quad (1)$$

and similarly,

$$d_{13}(t)-d_{13}(t-1)=\left(C_{13}^{3}-\hat{c}_{13}\right)\frac{d_{13}(t-1)}{e_{13}(t-1)}+C_{13}^{2}\frac{d_{12}(t-1)}{e_{12}(t-1)}. \quad (2)$$

With the initial condition that $\boldsymbol{d}(t_0)=[d_{12}(t_0),d_{13}(t_0)]$, the solution of the above two equations is

$$\boldsymbol{d}(t)=\left[d_{12}(t_0)\left(\frac{t}{t_0}\right)^{\gamma_{1m}},d_{13}(t_0)\left(\frac{t}{t_0}\right)^{\gamma_{1m}}\right], \quad (3)$$

where,

$$\gamma_{1m}=\frac{\left(C_{13}^{3}-\hat{c}_{13}\right)e_{12}+\left(C_{12}^{2}-\hat{c}_{12}-\right)e_{13}+\sqrt{\Delta}}{2e_{12}e_{13}}, \quad (4)$$

and,

$$\begin{aligned}\Delta=&-4\left(\left(C_{13}^{3}-\hat{c}_{13}\right)\left(C_{12}^{2}-\hat{c}_{12}\right)-C_{12}^{3}C_{13}^{2}\right)e_{12}e_{13}\\&+\left(\left(C_{13}^{3}-\hat{c}_{13}\right)e_{12}-\left(C_{12}^{2}-\hat{c}_{12}\right)e_{13}\right)^{2}.\end{aligned}$$

Here, $e_{12}=\frac{e_{12}(t)}{t}$ is a constant.

By the same method we can obtain the expression of $\boldsymbol{d}(t)$ for nodes in V_2 and V_3 and thus we complete the proof. ■

The result in Theorem 1 has the following two implications:

i) The degree $\boldsymbol{d}(t)$ grows with time t following a polynomial rate with power $\gamma_{km}\in(0,1)$.
ii) The two components of vector $\boldsymbol{d}(t)$ are in the same order, i.e., $d_{ki}(t)=\Theta\left(d_{kj}(t)\right)$.

The first implication gives the growth rate of node degree. Recall that $d_{ki}(t)$ denotes the degree of node v in $B(V_k,V_i)$ and $d_{kj}(t)$ denotes that in $B(V_k,V_j)$. The second one implies that the degrees of a certain node in two corresponding bipartite graphs appear to be approximately the same, which indicates that an important node (the node with larger degree) also behaves influential in all the other bipartite relationship networks and vice visa. Considering the example of academic networks, it is a common result that an author with more research topics is much likely to publish more papers. Consequently, the degree of it in authors-topics networks and that in authors-papers networks are in the same order.

*2) **Result 2: Degree distribution**:*

Based on the results in Theorem 1, we now come to analyze the degree distribution of nodes in $T(V_1,V_2,V_3)$.

Theorem 2: For the tripartite graph $T(V_1,V_2,V_3)$ generated after t time slots, when $t\to\infty$, the degree sequences of $v\in V_1$ of community C_m in $B\left(V_1,V_2\right)$ and $B\left(V_1,V_3\right)$ both follow a power-law distribution that

$$\mathbb{P}\left\{\boldsymbol{d}(t)=x\right\}=\frac{x^{-\frac{1}{\gamma_{1m}}-1}}{G_1},$$

where G_1 is a constant normalization coefficient. This result also holds for node $v\in V_2$ and $v\in V_3$ with a symmetrical expression.

Proof: For simplicity we first consider the distribution of $d_{12}(t)$, which denotes the degree of node $v\in V_1$ in $B(V_1,V_2)$. According to Equation (3), the Cumulative Distribution Function (CDF) of $d_{12}(t)$ can be calculated as

$$\begin{aligned}\mathbb{P}\left\{d_{12}(t)<x\right\}=&\mathbb{P}\left\{d_{12}(t_0)\left(\frac{t}{t_0}\right)^{\gamma_{1m}}<x\right\}\\=&\mathbb{P}\left\{t_0>t\left(\frac{d_{12}(t_0)}{x}\right)^{\frac{1}{\gamma_{1m}}}\right\}\\=&1-d_{12}(t_0)^{\frac{1}{\gamma_{1m}}}x^{-\frac{1}{\gamma_{1m}}},\end{aligned}$$

where the third equality holds since nodes are added to V_1 homogeneously with time t. Then, the Probability Density Function (PDF) of $d_{12}(t)$ can be calculated using $\mathbb{P}\left\{d_{12}(t)=x\right\}=\frac{\partial\mathbb{P}\{d_{12}(t)<x\}}{\partial x}$. Alternatively, we can also express it as

$$\mathbb{P}\left\{d_{12}(t)=x\right\}=\frac{x^{-\frac{1}{\gamma_{1m}}-1}}{G_1},$$

where $G_1=\sum_{x=1}^{n}x^{-\frac{1}{\gamma_{1m}}-1}$ is a constant normalization coefficient. By the same method, we can obtain the distributions of $d_{ij}(t)$, $i\neq j\in\{1,2,3\}$, which completes the proof. ■

The results in Theorem 2 are two folded:

i) The degree $\boldsymbol{d}(t)$ follows the power-law distribution with exponent $-\frac{1}{\gamma_{km}}-1$.
ii) The degree distributions of node v in different bipartite graphs have the same exponent.

The first result shows that our model can well capture the power-law distribution of node degrees. As for the similarity of degree distributions, the reason behind this result is the same with that in growth of node degree.

*3) **Result 3: Densification and shrinking diameter**:*

Theorem 3: In the tripartite graph $T(V_1,V_2,V_3)$ generated after t time slots, the ratio of edges to nodes in $G\left(V_i|V_j\right)$ is

$$\frac{|E|}{|V_i|}=\begin{cases}\Theta\left(t^{2-\frac{1}{\gamma_j}}\right), & \frac{1}{2}<\gamma_j<1\\ \Theta\left(\log t\right), & \gamma_j=\frac{1}{2}\\ \Theta\left(1\right), & 0<\gamma_j<\frac{1}{2},\end{cases}$$

where $\gamma_j=\max_m\left\{\gamma_{jm}\right\}$.

Proof: According to the definition of $G\left(V_i|V_j\right)$, each node $v\in V_j$ in $B\left(V_i,V_j\right)$ gives rise to a clique where all neighbors of v are connected. Thus, the average number of edges in $G\left(V_i|V_j\right)$ that are resulted by the nodes of community C_m in V_j is

$$|E_m|=\sum_{k=1}^{\alpha_i t}\alpha_j\beta_m t\cdot\frac{k^{-\frac{1}{\gamma_{jm}}-1}}{G_j}\binom{k}{2},$$

where $\alpha_j\beta_m t$ is the expected number of nodes of community C_m in V_j. The probability that the node's degree equals to k is $\frac{k^{-\frac{1}{\gamma_{jm}}-1}}{G_j}$, and each of them gives rise to $\binom{k}{2}$ edges in $G\left(V_i|V_j\right)$. Given the sum of *p-series*

$$\lim_{n\to\infty}\sum_{x=1}^{n}\frac{1}{x^p}=\begin{cases}\Theta\left(n^{1-p}\right), & 0\leq p<1\\ \Theta\left(\log n\right), & p=1\\ \Theta\left(1\right), & p>1,\end{cases}$$

we have

$$|E_m| = \begin{cases} \Theta\left(t^{3-\frac{1}{\gamma_{jm}}}\right), & \frac{1}{2} < \gamma_{jm} < 1 \\ \Theta\left(t \log t\right), & \gamma_{jm} = \frac{1}{2} \\ \Theta\left(t\right), & 0 < \gamma_{jm} < \frac{1}{2}. \end{cases}$$

Summing over all possible m and noting that $|V_i| = \alpha_i t$, we obtain the final results ∎

We proceed to present that in our model, network diameter shrinks or stabilizes over time. Before the proof we first give a useful result shown in Lemma 1.

Lemma 1: In $B(V_i, V_j)$, the probability that an edge points to a node $j \in V_j$ of community C_m with degree x is

$$\mathbb{P}\left\{d_j(t) = x\right\} = \frac{x^{-\frac{1}{\gamma_{jm}}}}{\hat{G}_j},$$

where $\hat{G}_j$ is a constant normalization coefficient.

Proof: According to Theorem 2, the probability that a node $j \in V_j$ in $B(V_i, V_j)$ has degree x is

$$\mathbb{P}\left\{d_j = x\right\} \propto x^{-\frac{1}{\gamma_{jm}}-1}.$$

The number of nodes with degree x is $n\mathbb{P}\left\{d_j = x\right\}$ and the probability that an edge in $B(V_i, V_j)$ points to one of these nodes is $\frac{xn\mathbb{P}\{d_j=x\}}{e_{ij}(t)}$. Alternatively, we can also express it as $\frac{x^{-\frac{1}{\gamma_{jm}}}}{\hat{G}_j}$ with normalization coefficient $\hat{G}_j = \sum_{x=1}^{n} x^{-\frac{1}{\gamma_{jm}}}$. ∎

We use the definition of q-effective diameter given in [28] to measure the network diameter.

Definition 1: (q-effective diameter) q-effective diameter is defined as the minimum d such that, for at least a q fraction of nodes, the shortest path between any two nodes is at most d, $0 < q < 1$.

Theorem 4: In the generated graph $G\left(V_i|V_j\right)$, if $\gamma_j = \max_m\left\{\gamma_{jm}\right\} \geq \frac{1}{2}$, the q-effective diameter shrinks or stabilizes at time $t \geq \phi n$, for any constants $\phi > 0, 0 < q < 1$.

Proof: Let C denote the set of nodes of V_j in $B(V_i, V_j)$ with degree $d_{ij} \geq n^{\sigma}$, where $\sigma > 0$ is a small constant. By Theorem 1 we have, every node in C arrives before time εn, $\varepsilon > 0$. Thus the diameter of the neighborhood of C in $G\left(V_i|V_j\right)$, i.e., $dia(C)$, shrinks or stabilizes after time $t = \phi n$.

Now we show that all but only $o(n)$ nodes added after time $t = \phi n$ have at least a friend that is the neighborhood of C. According to Lemma 1, if $\gamma_j = \max_m\left\{\gamma_{jm}\right\} \geq \frac{1}{2}$, the number of edges in $B(V_i, V_j)$ whose endpoints are not nodes in C can be upper bounded by

$$\sum_m \frac{1}{\hat{G}_j} \sum_{x=1}^{n^{\sigma}} x \cdot x^{-\frac{1}{\gamma_{jm}}} \leq M \frac{1}{\hat{G}_j} \sum_{x=1}^{n^{\sigma}} x \cdot x^{-\frac{1}{\gamma_j}} = o(n).$$

Hence the q-effective diameter of $G\left(V_i|V_j\right)$ is upper bounded by $dia(C) + 2$ and we complete the proof. ∎

The above results imply that, the generated graph $G(V_i|V_j)$ densifies and its diameter shrinks or stabilizes with time t if it is satisfied that $\gamma_j = \max_m\left\{\gamma_{jm}\right\} \geq \frac{1}{2}$.

B. Properties on Community Structure

Besides the aforementioned network properties, our model can also incorporate community structure, which is our focus in this part. In doing so, we first introduce some preliminaries on community and its evolution, and then present our results.

*1) **Result 4: Community structure**:*

Quantitive metrics of community structure vary a lot and in our work, we use community ratio, as defined below, to make the measurement.

Definition 2: (Internal and External Degree) The internal and external degree of community C_m, denoted by k_{in}^m and k_{ex}^m, are defined as the sum of degrees of node $v \in C_m$ connecting to node of the same community and the sum of that connecting to the rest of nodes, respectively.

Definition 3: (Community Ratio) We define the total degrees of community C_m, denoted by k^m, as the number of degrees of node in community C_m. Then, the intra-community ratio, denoted by D_{in}^m, and the inter-community ratio, denoted by D_{ex}^m, are defined as

$$D_{in}^m = \frac{k_{in}^m}{k^m} \quad \text{and} \quad D_{ex}^m = \frac{k_{ex}^m}{k^m},$$

respectively. By the definition, we know $D_{in}^m, D_{ex}^m \in [0, 1]$ and $D_{in}^m + D_{ex}^m = 1$. Since D_{ex}^m can be calculated through $D_{ex}^m = 1 - D_{in}^m$, in the following analysis we only take D_{in}^m into consideration. Under the given definition, the network can be classified into three types:

- *Community Structure:* $D_{in}^m > \frac{1}{2}$. The ratio of internal degrees is greater than that of external degrees.
- *Random Structure:* $D_{in}^m = \frac{1}{2}$. The ratio of internal degrees is equal to that of external degrees.
- *Disassortative Structure:* $D_{in}^m < \frac{1}{2}$. The ratio of internal degrees is less than that of external degrees.

Firstly, we give a Theorem to show that the network in our model exhibits community structure.

Theorem 5: In Evolving 3-Graph generated after t time slots, if the community probability $p > \frac{2}{3}$, the nodes in generated graph $G\left(V_i|V_j\right)$ are community-structured, that is,

$$D_{in}^m > \frac{1}{2}.$$

Proof: Firstly, let us come to calculate the number of total degrees of an arbitrary community C_m, that is,

$$k^m = \sum_{v_j \in C_m} \sum_k \left(\binom{k}{2} - \binom{(1-p)k}{2} \right) \frac{k^{-\frac{1}{\gamma_{jm}}-1}}{G_j} + \sum_{n \neq m} \sum_{v_j \in C_n} \sum_k \left(\binom{k}{2} - \binom{k - k(1-p)\frac{\beta_m}{1-\beta_n}}{2} \right) \frac{k^{-\frac{1}{\gamma_{jn}}-1}}{G_j},$$

where v_j is the node in set V_j and we make the calculation by considering v_j in different communities. The first item in the Equation denotes the expected number of edges in k^m that resulted by the v_j of community C_m. Note that these part of nodes that similarly belong to community C_m in V_j have degree k with probability $\mathbb{P}\left\{d_{ji}(t) = k\right\} = \frac{x^{-\frac{1}{\gamma_{jm}}-1}}{G_j}$ according to Theorem 2. Each of these nodes results in a clique in $G\left(V_i|V_j\right)$ with $\binom{k}{2}$ total edges. Except $\binom{(1-p)k}{2}$ edges

connecting to nodes in V_i of other communities, all the other edges connecting to at least one node of community C_m in V_i. On the other hand, the second item in the Equation denotes the expected number of edges in k^m that resulted by the v_j of community C_n in V_j, where $n \neq m$. In this part, only $k(1-p)\frac{\beta_m}{1-\beta_n}$ edges connecting to node $v_i \in C_m$ in V_i and the corresponding edges in $G\left(V_i|V_j\right)$ can be calculated as in the Equation. Then, summing these two items together we obtain k^m. Since that the second item is greatly smaller than the first one in order sense, the above Equation can be simplified to

$$k^m = \sum_{v_j \in C_m} \sum_{k} \left(\binom{k}{2} - \binom{(1-p)k}{2} \right) \frac{k^{-\frac{1}{\gamma_{jm}}-1}}{G_j}. \quad (5)$$

Secondly, the internal degree of community C_m is

$$k_{in}^m > \sum_{v_j \in C_m} \sum_{k} \binom{pk}{2} \frac{k^{-\frac{1}{\gamma_{jm}}-1}}{G_j}. \quad (6)$$

Essentially, the internal degree of community C_m is also composed of two parts: edges that resulted by v_j of community C_m and that of other communities. For simplicity, we neglect the latter one and thus obtain Equation (6).

Combining Equation (5) and Equation (6), we have

$$D_{in}^m = \frac{k_{in}^m}{k^m} > \frac{p^2}{1-(1-p)^2}.$$

With the condition $p > \frac{2}{3}$, we complete the proof. ■

The above results have the following two implications:

i) The Evolving K-graph exhibits community structure.
ii) The intra-community ratio, i.e., D_{in}^m, is a constant that remains stable in the whole evolving process.

The first result implies that Evolving K-Graph can well capture the community structure existed in real networks. The second result indicates that community structure often remains stable during network evolution unless there are occurrences of some events such as merge or split, which we will discuss then.

*2) **Result 5: Merge and split**:*

As two basic events of community evolution, we formulate merge and split in our model as follows:

– *Merge:* $C_1 + C_2 \to C_3$. Communities C_1 and C_2 start to merge into a new community C_3 at time t_0 by removing β_1 and β_2 from the community control set and adding a new one $\beta_3 = \beta_1 + \beta_2$ instead.
– *Split:* $C_3 \to C_1 + C_2$. A community C_3 starts to split into communities C_1 and C_2 at time t_0 by removing β_3 from the community control set and adding two new ones β_1 and β_2 that satisfy $\beta_1 + \beta_2 = \beta_3$ instead.

The merge and split of community are measured as follows.

Definition 4: (Merging and Splitting Time) The merging time, denoted by Δt_m, of the event $C_1 + C_2 \to C_3$ is defined as the time period that starts at the beginning time of merge event, i.e., t_0, and ends after Δt_m time slots with a stable D_{in}^3. The splitting time, denoted by Δt_s, of the event $C_3 \to C_1 + C_2$ is defined in a similar way that ends when both D_{in}^1 and D_{in}^2 become stable.

Theorem 6: In Evolving 3-Graph, the merging time of the event $C_1 + C_2 \to C_3$ satisfies

$$\Delta t_m = \omega(t_0),$$

where t_0 is the beginning time of the merge event. And similarly, the splitting time of the event $C_3 \to C_1 + C_2$ satisfies

$$\Delta t_s = \omega(t_0),$$

where t_0 is the beginning time of the split event.

Proof: Let us first consider the merging time of the event $C_1 + C_2 \to C_3$. Note that at the end of merge event, the network achieves a new stable state and thus the network properties we have given hold. Considering the generated graph $G(V_i|V_j)$, according to Theorem 2 we have that the degree of node $v \in C_3$ in V_j satisfies

$$\mathbb{P}\left\{d(t) = x\right\} = \frac{x^{-\frac{1}{\gamma_{j3}}-1}}{G_j}.$$

According to Equations (5) and (6), in order to obtain the intra-community ratio D_{in}^m, we should also calculate the fraction of edges of nodes in V_j that connect to nodes in V_i of community C_m.

For an arbitrary node $v \in C_3$ that is added to V_j before time t_0, the number of its edges connecting to nodes of community C_3 in V_i among its total k edges is

$$\begin{aligned}\tilde{p}k = &\frac{\Delta t_m}{\Delta t_m + t_0} kp + \\ &\frac{t_0}{\Delta t_m + t_0} k \frac{\beta_1}{\beta_1 + \beta_2} \left(p + (1-p)\frac{\beta_2}{1-\beta_1} \right) + \\ &\frac{t_0}{\Delta t_m + t_0} k \frac{\beta_2}{\beta_1 + \beta_2} \left(p + (1-p)\frac{\beta_1}{1-\beta_2} \right).\end{aligned}$$

Note that $\frac{\Delta t_m}{\Delta t_m + t_0} k$ edges are added after time t_0 following the new rule and thus p fraction of them connect to nodes of community C_3 in V_i, which is given in the first item of the Equation. Then, the other $\frac{t_0}{\Delta t_m + t_0} k$ edges are added before time t_0. The corresponding node in V_j originally belongs to C_1 with probability $\frac{\beta_1}{\beta_1+\beta_2}$ and in this case, p of its edges connecting to nodes $v \in C_3$ in V_i that originally belong to C_1 and $(1-p)\frac{\beta_2}{1-\beta_1}$ of them connecting to those originally belonging to C_2, as given in the second item of the equation. The third item of the equation is calculated similarly for node in V_j that originally belongs to C_2. In addition, nodes added after time t_0 follow the results given in Theorem 5.

Finally, according to Equations (5) and (6) we know that only if $\tilde{p} \to p$ the merge event ends. This required condition holds when $\Delta t_m = \omega(t_0)$ and we obtain the result. The proof of the split event follows a similar method and thus we omit it here for concision. ■

This result indicates that once an event happens, the network needs a considerable time to recover and achieve a new stable state. While, we note that since most of community detection algorithms [29], [30], [31] work by maximizing "modularity" [32], the community partition result will be updated before the event ends, which thus does not require such a long time.

C. Properties on Correlation and Network Connectivity

All the network properties we discussed are common ones in single-typed evolving networks. In this part, we study some distinctive ones that only exists in multi-typed networks.

*1) **Result 6: Positive correlation**:*

For an Evolving 3-Graph, the connections among nodes in V_k can be classified into two types:

- *Type 1:* Connections in $G(V_k|V_i)$ generated from $B(V_k, V_i)$.
- *Type 2:* Connections in $G(V_k|V_j)$ generated from $B(V_k, V_j)$.

Therefore, the generated network of nodes in V_k is two-layered, and we denote it as $\boldsymbol{G}(V_k) = [G(V_k|V_i), G(V_k|V_j)]$. Based on the above network structure, we measure the properties of generated network from the following two aspects.

For a two-layered network, the correlation of the node degrees in two layers can be measured by degree correlation coefficient, which is defined as follows:

Definition 5: (Degree Correlation Coefficient) The degree correlation coefficient of a two-layered network is defined as

$$\rho(d_1, d_2) = \frac{\mathbb{E}[d_1 d_2] - \mathbb{E}[d_1]\mathbb{E}[d_2]}{\sqrt{\mathbb{E}\left[{d_1}^2\right] - \mathbb{E}[d_1]^2}\sqrt{\mathbb{E}\left[{d_2}^2\right] - \mathbb{E}[d_2]^2}},$$

where d_1 is the degree in layer 1 and d_2 is that in layer 2. According to degree correlation coefficient, networks can be classified into the following three types:

– *Positively-Correlated:* $\rho(d_1, d_2) > 0$. Node with large degree in layer 1 also has large degree in layer 2.
– *UnCorrelated:* $\rho(d_1, d_2) = 0$. The degree of a node in layer 1 has no relation with that in layer 2.
– *Negatively-Correlated:* $\rho(d_1, d_2) < 0$. Node with large degree in layer 1 has small degree in layer 2.

To focus on correlation and connectivity, in the remaining part we remove the community structure by setting the parameters as $\{\beta_1 = 1\}$ and as $p = 1$.

Theorem 7: In the generated network at time t, when $t \to \infty$, the degree of node $v \in V_1$ in $G(V_1|V_2)$, denoted by d_{12}, and that in $G(V_1|V_3)$, denoted by d_{13}, are positively-correlated, that is,

$$\rho(d_{12}, d_{13}) > 0.$$

This result also holds for node $v \in V_2$ and $v \in V_3$ with a symmetrical expression.

Proof: According to Definition 5, we need to prove

$$\mathbb{E}[d_{12}d_{13}] > \mathbb{E}[d_{12}]\mathbb{E}[d_{13}].$$

In the following, for convenience, we use several simplified notations to denote the degrees in $T(V_1, V_2, V_3)$. Let d_2 denote the degree of node $v \in V_1$ in $B(V_1, V_2)$, d_3 denote that in $B(V_1, V_3)$, i denote the degree of node in V_2 that an arbitrary edge in $B(V_1, V_2)$ points to and j denote the degree of node in V_3 that an arbitrary edge in $B(V_1, V_3)$ points to. The distributions of these four random variables can be obtained from Theorem 2 and Lemma 1.

Firstly, let's consider the average degree of node $v \in V_1$ in $G(V_1|V_2)$. According to the generating rule, the degree of v in $G(V_1|V_2)$ can be considered as the sum of degrees of the node in V_2 that the edges of v in $B(V_1, V_2)$ lead to, that is,

$$\mathbb{E}[d_{12}] = \mathbb{E}\left[\sum_{k=1}^{d_2} i_k\right] = \mathbb{E}[d_2]\mathbb{E}[i], \tag{7}$$

where the second equality holds since i_k is an independent and identically distributed random variable. Then, the average degree of node $v \in V_1$ in $G(V_1|V_3)$ can be obtained similarly, that is,

$$\mathbb{E}[d_{13}] = \mathbb{E}\left[\sum_{k=1}^{d_3} j_k\right] = \mathbb{E}[d_3]\mathbb{E}[j]. \tag{8}$$

Secondly, $\mathbb{E}[d_{12}d_{13}]$ can be calculated following the same method

$$\mathbb{E}[d_{12}d_{13}] = \mathbb{E}\left[\sum_{k=1}^{d_2} i_k \cdot \sum_{k=1}^{d_3} j_k\right] = \mathbb{E}[i]\mathbb{E}[j]\mathbb{E}[d_2 d_3]. \tag{9}$$

Based on Equation (7), Equation (8) and Equation (9), we have

$$\mathbb{E}[d_{12}d_{13}] - \mathbb{E}[d_{12}]\mathbb{E}[d_{13}] = \mathbb{E}[i]\mathbb{E}[j]\left(\mathbb{E}[d_2 d_3] - \mathbb{E}[d_2]\mathbb{E}[d_3]\right).$$

Then according to Theorem 1 we know

$$\frac{d_2}{d_{12}(t_0)} = \frac{d_3}{d_{13}(t_0)}. \tag{10}$$

Equation (10) indicates that d_2 and d_3 are positively-correlated and thus we complete the proof. ■

We note that this result can be widely observed in realistic networks, which indicates that a user with many friends in a certain network is also influential in another one. Moreover, this result is also verified in the experimental observations on real datasets in Section III.

*2) **Result 7: Reduced phase transition time**:*

In addition to the correlation coefficient, we define the other metric named *Phase Transition Time* to evaluate the properties of network connectivity in evolving networks.

Definition 6: (Phase Transition Time) In evolving networks, the metric phase transition time t_p is defined as the threshold time when the giant component emerges, that is, if $t < t_p$,

$$C = o(n),$$

and if $t \geq t_p$,

$$C = \Theta(n),$$

where C is the size of giant component and n is the total number of nodes in the network.

Obviously, a smaller *Phase Transition Time* indicates a better network connectivity in evolving networks, which can further lead to a better performance of information cascading, network robustness, resource prediction, system behavior, etc. To better demonstrate the performance of Evolving K-Graph, we present a comparison between our model and Multi-typed Affiliation Network. The result is given in Theorem 8.

Lemma 2: The value of parameter γ_{im} lies in the range $\gamma_{im} \in (0, 1)$. Moreover, any two of parameters γ_{1m}, γ_{2m} and γ_{3m} can not be smaller than $\frac{1}{2}$ at the same time.

Proof: The conclusion that $\gamma_{im} \in (0, 1)$ can be verified by Equation (4), and we focus on the second result in the following proof. Without loss of generality, we prove that the case $\gamma_{1m}, \gamma_{2m} < \frac{1}{2}$ does not exist. Note that the value of γ_{1m} (γ_{2m}) increases monotonically with α_2 and α_3 (α_1 and α_3), and thus in order to satisfy the case, α_3 should be set as small as possible. Letting $\alpha_3 \to 0$, we have

$$\gamma_{1m} = \frac{C_{12}^2}{C_{12}^1 + C_{12}^2} \quad \text{and} \quad \gamma_{2m} = \frac{C_{12}^1}{C_{12}^1 + C_{12}^2}.$$

Note that $\gamma_{1m} < \frac{1}{2}$ when $C_{12}^1 > C_{12}^2$ and $\gamma_{2m} < \frac{1}{2}$ when $C_{12}^1 < C_{12}^2$, which leads to a contradiction. Hence we conclude

that γ_{1m} and γ_{2m} can not be smaller than $\frac{1}{2}$ at the same time and we complete the proof. ■

Theorem 8: In the generated network $\boldsymbol{G}(V_k)$, $t \to \infty$, the phase transition time t_g is much smaller than that of Multi-typed Affiliation Network, denoted by t_b, which satisfies

$$t_g = \Theta\left({t_b}^c\right),$$

where $c \leq 1$ is a constant.

Proof: We make the calculation by using generating functions [33]. Let $h_2(x)$ denote the generating function of the size of component reached by following an edge in $G(V_1|V_2)$ and let $h_3(x)$ denote that in $G(V_1|V_3)$. In other words, we define $h_2(x)$ as

$$h_2(x) = \sum_k p_k x^k,$$

where p_k is the probability that an arbitrary edge in $G(V_1|V_2)$ leads to a component of size k. And $h_3(x)$ is defined in a same way. The self-consistency conditions of $h_2(x)$ and $h_3(x)$ are

$$h_2(x) = x \sum_{\boldsymbol{d}} \frac{d_v p_{\boldsymbol{d}}}{\mathbb{E}[d_v]} h_2(x)^{d_v - 1} h_3(x)^{d_u}$$

$$h_3(x) = x \sum_{\boldsymbol{d}} \frac{d_u p_{\boldsymbol{d}}}{\mathbb{E}[d_u]} h_2(x)^{d_v} h_3(x)^{d_u - 1},$$

where $\boldsymbol{d} = [d_v, d_u]$ is the node degree in generated network. Let $H(x)$ denote the generating function of size of component reached by following an arbitrary edge. Using the same method we have

$$H(x) = x \sum_{\boldsymbol{d}} p_{\boldsymbol{d}} h_2(x)^{d_v} h_3(x)^{d_u}.$$

This equation holds since a node in generated network can be reached either by an edge in $G(V_1|V_2)$ or an edge in $G(V_1|V_3)$. The Jacobian matrix $\boldsymbol{J}$ of the equations can be generated as

$$\boldsymbol{J} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} \frac{\mathbb{E}[{d_v}^2 - d_v]}{\mathbb{E}[d_v]} & \frac{\mathbb{E}[d_v d_u]}{\mathbb{E}[d_v]} \\ \frac{\mathbb{E}[d_v d_u]}{\mathbb{E}[d_u]} & \frac{\mathbb{E}[{d_u}^2 - d_u]}{\mathbb{E}[d_u]} \end{bmatrix}.$$

Phase transition point exists when the eigenvalues of $\boldsymbol{J}$ equal to 1

$$\sigma\left(\boldsymbol{J}\right) = \frac{1}{2}\left(a_{11} + a_{22} + \sqrt{\left(a_{11} - a_{22}\right)^2 + 4a_{12}a_{21}}\right) = 1.$$

To derive the phase transition point, we first need to calculate the items in the Jacobian matrix $\boldsymbol{J}$.

1) $\mathbb{E}[d_v]$ and $\mathbb{E}[d_u]$: According to Equations (7) and (8), we have

$$\mathbb{E}\left[d_v\right] = \mathbb{E}\left[d_2\right]\mathbb{E}\left[i\right] = \Theta\left(\sum_{k=1}^{n} k^{1-\frac{1}{\gamma_2}}\right)$$

$$\mathbb{E}\left[d_u\right] = \mathbb{E}\left[d_3\right]\mathbb{E}\left[j\right] = \Theta\left(\sum_{k=1}^{n} k^{1-\frac{1}{\gamma_3}}\right).$$

The probability distributions of d_2 and d_3 are given in Theorem 2, and that of i and j are given in Lemma 1, based on which we can obtain the above results.

2) $\mathbb{E}[{d_v}^2]$ and $\mathbb{E}[{d_u}^2]$: Since $d_v = \sum_{k=1}^{d_2} i_k$, we have

$$\begin{aligned}\mathbb{E}[{d_v}^2] &= \mathbb{E}[d_2]\mathbb{E}[i^2] + \mathbb{E}[d_2]\left(\mathbb{E}[d_2] - 1\right)\mathbb{E}[i]^2 \\ &= \Theta\left(\sum_{k=1}^{n} k^{2-\frac{1}{\gamma_2}}\right),\end{aligned}$$

where $\mathbb{E}[d_2] = \Theta(1)$ and $\mathbb{E}[i^2] \geq \mathbb{E}[i]^2$. And similarly, we have

$$\mathbb{E}[{d_u}^2] = \Theta\left(\sum_{k=1}^{n} k^{2-\frac{1}{\gamma_3}}\right).$$

3) $\mathbb{E}\left[d_v d_u\right]$: According to Equation (9), we have

$$\begin{aligned}\mathbb{E}[d_v d_u] =& \mathbb{E}\left[i\right]\mathbb{E}\left[j\right]\mathbb{E}\left[d_2 d_3\right] = \mathbb{E}[d_v]\mathbb{E}[d_u]\frac{\mathbb{E}[d_2 d_3]}{\mathbb{E}[d_2]\mathbb{E}[d_3]} \\ =& \mathbb{E}[d_v]\mathbb{E}[d_u]\Theta\left(\sum_{k=1}^{n} k^{1-\frac{1}{\gamma_1}}\right).\end{aligned} \tag{11}$$

Note that the expression of $\mathbb{E}[d_v d_u]$ in our model is given in Equation (11) since d_v and d_u are positively-correlated. However, in the Multi-typed Affiliation Network we have $\mathbb{E}[d_v d_u] = \mathbb{E}[d_v]\mathbb{E}[d_u]$.

The above items vary with parameters γ_1, γ_2 and γ_3, and thus we divide this problem into several cases.

Case 1: $\gamma_1 \in (0, \frac{1}{2}]$. In this case, according to Equation (11) we know that for the generated network

$$\mathbb{E}[d_v d_u] = \Theta\left(\mathbb{E}[d_v]\mathbb{E}[d_u]\right).$$

Therefore, we have

$$t_g = \Theta\left(t_b\right).$$

Case 2: $\gamma_1 \in (\frac{1}{2}, 1)$ and $\gamma_2, \gamma_3 \in (0, \frac{1}{2})$. According to Lemma 2 we know that this case does not exist.

Case 3: $\gamma_1, \gamma_2, \gamma_3 \in (\frac{1}{2}, 1)$. For the generated network, we have

$$\sqrt{{n_g}^2 + 4{n_g}^{4-\frac{1}{\gamma_2}-\frac{1}{\gamma_3}}} - n_g = \Theta(1). \tag{12}$$

For the one obtained from Multi-typed Affiliation Network, we have

$$\sqrt{{n_b}^2 + 4{n_b}^{8-\frac{1}{\gamma_2}-\frac{1}{\gamma_3}-\frac{2}{\gamma_1}}} - n_b = \Theta(1). \tag{13}$$

Note that in evolving networks, $n = \Theta(t)$. Combining the Equations (12) and (13), we can obtain

$$t_g = \Theta\left(t_b^{\frac{2}{8-\frac{1}{\gamma_2}-\frac{1}{\gamma_3}-\frac{2}{\gamma_1}}}\right),$$

if $8 - \frac{1}{\gamma_2} - \frac{1}{\gamma_3} - \frac{2}{\gamma_1} > 2$; otherwise,

$$t_g = \Theta\left(t_b\right).$$

Case 4: $\gamma_1, \gamma_2 \in (\frac{1}{2}, 1)$ and $\gamma_3 \in (0, \frac{1}{2})$. In this case, we have $\frac{\mathbb{E}[{d_v}^2 - d_v]}{\mathbb{E}[d_v]} \geq \frac{\mathbb{E}[{d_u}^2 - d_u]}{\mathbb{E}[d_u]}$ and thus the phase transition time can be calculated as follows. For the generated network, we have

$$\sqrt{{n_g}^2 + 4{n_g}^{2-\frac{1}{\gamma_2}}} - n_g = \Theta(1).$$

For the one generated from Multi-typed Affiliation Network, we have

$$\sqrt{{n_b}^2 + 4{n_b}^{6-\frac{1}{\gamma_2}-\frac{2}{\gamma_1}}} - n_b = \Theta(1).$$

Combining the the above two equations, we can obtain

$$t_g = \Theta\left(t_b^{\frac{2}{6-\frac{1}{\gamma_2}-\frac{2}{\gamma_1}}}\right),$$

if $6 - \frac{1}{\gamma_2} - \frac{2}{\gamma_1} > 2$; otherwise,

$$t_g = \Theta\left(t_b\right).$$

Case 5: $\gamma_1, \gamma_3 \in (\frac{1}{2}, 1)$ and $\gamma_2 \in (0, \frac{1}{2})$. The result can be obtained with a same method as in case 4. ■

Remark: Although results in Theorem 8 indicate that the phase transition time of generated network is equal ($c = 1$) or smaller ($c < 1$) than that in Multi-typed Affiliation Network in order sense. We will show that the case $c < 1$ exists in most conditions. According to Theorem 3 and Theorem 4 we know that, the densification and shrinking diameter exist in $G(V_i|V_j)$ if $\gamma_j \geq \frac{1}{2}$ and a larger γ_j indicates a faster speed. Therefore, we can infer that in most realistic networks, the condition $\gamma_j \geq \frac{1}{2}$ holds for $j \in \{1, 2, 3\}$. Then, according to Theorem 8, most of networks can be classified into Case 3 with the large enough parameters, and consequently,

$$t_p = \Theta\left(t_b{}^c\right),$$

where $c < 1$ holds in most realistic networks.

Thanks to the interactive-evolution, we observe a relatively smaller phase transition time in the proposed model compared with Multi-typed Affiliation Networks, which gives an earlier occurrence of connection among a constant fraction of nodes.

D. Extension to Evolving K-Graph

In this subsection, we extend the Evolving 3-Graph model to the general case $K \geq 3$. As expected, the main results in Evolving 3-Graph also hold in the extended one. Specifically, for an arbitrary node $v \in V_k$ in $T(V_1, ..., V_K)$, it satisfies

- *Growth of node degree:* The degree of node v increases with time t, i.e., $\boldsymbol{d}(t) = \Theta\left(t^{\gamma_{km}}\right)$, where $\gamma_{km} \in (0, 1)$ is a constant. Moreover, all components of vector $\boldsymbol{d}(t)$ are in the same order.
- *Degree distribution:* The degree of node v follows the power-law distribution with exponent $-\frac{1}{\gamma_k} - 1$.
- *Densification and shrinking diameter:* The generated graph $G(V_k|V_i)$, $k \neq i$, has the properties of densification and shrinking diameter if $\gamma_i \geq \frac{1}{2}$.
- *Community structure:* The intra-community ratio D_{in}^m remains stable with the network evolution and requires $\omega(t_0)$ time to recover after an event.
- *Degree correlation:* For the generated network $\boldsymbol{G}(V_k)$, the degree correlation coefficient of any two layers is positive.
- *Phase transition time:* The phase transition time of generated network $\boldsymbol{G}(V_k)$ is much smaller than that without *interactive-evolution* in order sense.

To prove the above results, let us consider how the node degree grows in each time slot. Following the same method as in Theorem 1, the degree of $v \in V_k$ in $B(V_k, V_i)$ satisfies

$$d_{ki}(t) - d_{ki}(t-1) = C_{ki}^i \frac{d_{ki}(t-1)}{e_{ki}(t-1)} + \sum_{j \neq k,i} C_{ki}^j \frac{d_{kj}(t-1)}{e_{kj}(t-1)},$$

where d_{ki} is the degree of $v \in V_k$ in $B(V_k, V_i)$. As we can observe, interactive-evolution (the second part in the equation) dominates the growth of node degree, which further leads to other results following the same way as in Evolving 3-Graph.

TABLE IV
PARAMETERS IN FITTING OF CONTACT NETWORK

Parameters	Evolving K-Graph	Multi-typed Affiliation Network
$< c_{ij} >$	$< 5 >$	$< 9 >$
$< \hat{c}_{ij} >$	$< 0.25 >$	$< 0.25 >$
$< p_1, p_2, p_3 >$	$< 0.29, 0.38, 0.43 >$	$< 0.29, 0.38, 0.43 >$

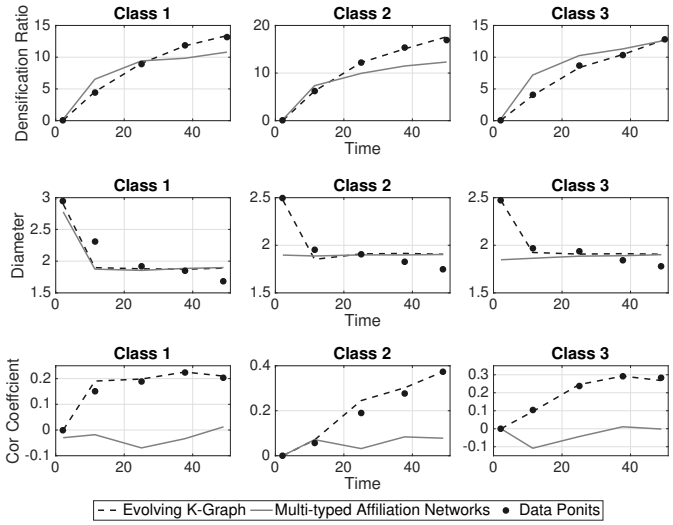


Fig. 8. Fitting of the Contact Network dataset, where Evolving K-Graph and Multi-typed Affiliation Networks are applied to reproduce three main features, i.e., densification, shrinking diameter and positive correlation.

VI. SIMULATIONS

This section gives experiments of fitting Evolving K-Graph model to real-world networks. Given a particular dataset, we aim to reproduce its main features through our proposed model as well as other baseline models for performance comparison.

A. Simulation Settings

1) Dataset: The dataset used in simulations is the Contact Network, the details of which could be found in Section III-A.

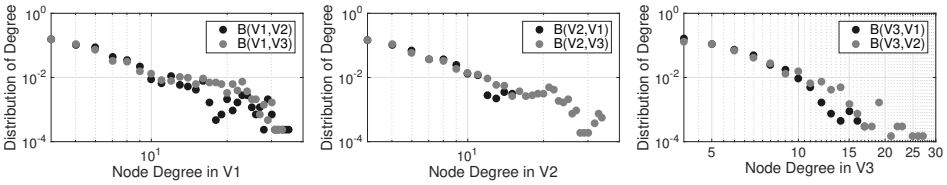
2) Model used in Comparison: As illustrated in Section II, to our best knowledge, there are no existing models designed for multi-typed evolving networks; Instead, some models for single-typed ones can be simply modified to a multi-typed one and thus we include one of them for performance comparison.

- **Multi-typed Affiliation Network:** A modification of Affiliation Network [15], where at each time slot, a certain typed node arrives with a given probability and attaches to nodes of other types following a preferential attachment manner.

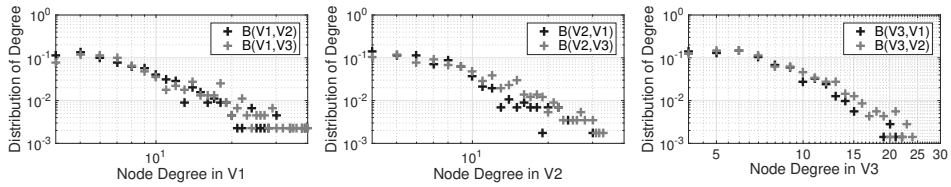
3) Parameter Settings: The values of parameters applied in fitting are listed in Table IV. The objective of fitting is to look for a group of parameters that minimizes the sum of relative error, between simulated results and real ones, over all the test points. The objective function we aim to minimize is

$$\min \sum\nolimits_{\text{feature } i} \sum\nolimits_{\text{test point } j} \left(h_{ij} - d_{ij}\right)^2,$$

where h_{ij} and d_{ij} denote the simulated results and real ones, respectively. In the fitting process, we adjust parameters in the following way. Firstly, in both Evolving K-Graph and Multi-typed Affiliation Network, $< p_1, p_2, p_3 >$ directly determines



(a) Evolving K-Graph.



(b) Multi-typed Affiliation Network.

Fig. 9. Degree distributions of bipartite graphs in Evolving K-graph and Multi-typed Affiliation Network with nodes types V_1, V_2 and V_3.

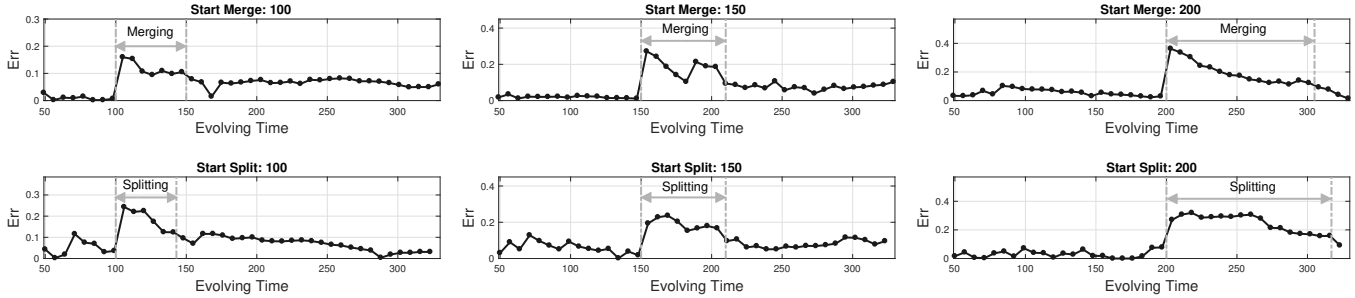


Fig. 10. Simulation results on community evolution. The top three subfigures show the merge event with start times $t = 100$, $t = 150$ and $t = 200$ respectively and the bottom three subfigures give the corresponding results of split event.

the number of nodes in the network, i.e., $\mathbb{E}[|V_i|] = p_i t$. Hence, we set it as $< 0.29, 0.38, 0.43 >$, which is proportional to the number of nodes in Contact Network. Secondly, in both two models, $< c_{ij} >$ and $< \hat{c}_{ij} >$ jointly determine the speed of network evolution, where a larger $< c_{ij} >$ leads to a faster evolution and a larger $< \hat{c}_{ij} >$ results in a slower one. For the convenience, we fix the value of $< \hat{c}_{ij} >$ and adjust $< c_{ij} >$ to make the fitting. We note that fix $< c_{ij} >$ and adjust $< \hat{c}_{ij} >$, or adjust the two parameters simultaneously can also achieve the goal. In addition, due to the different evolving mechanisms, performances of these two models, under the same parameters, differ with each other. Therefore, we conduct fitting of the two models separately and finally obtain the values as in Table IV.

B. Performance

1) Performance of Evolving K-Graph: We reproduce three main features, i.e., densification, shrinking diameter and positive correlation, and the corresponding results are given below.

Densification: With the given parameters, the network generated by Evolving K-Graph densifies as illustrated in Fig. 8. From the figure we can observe that the densification ratios at five data points, in all the three node types, well fit to that in the Contact Network dataset. In particular, the largest absolute error among all the 15 data points is only 0.3159.

Shrinking diameter: With the same parameters, diameter of the generated network shrinks with the time. The corresponding results are given in Fig. 8. We can find that the diameter of the dataset shrinks slowly while in our generated network, it first decreases rapidly and then remains stable. However, the fitness can still be regarded as a well performed one since the largest absolute error among all the data points is only 0.4074.

Positive correlation: In the generated network, node degrees of a certain type are positively correlated, which well fit to that in the Contact Network dataset under the given parameters, as we could observe from the results presented in Fig. 8.

In addition to the fitting in terms of the above three features, we present how our model produce the other two, i.e., power-law degree distribution and community evolution. We note that these two features are not included as targets in the fitting since the corresponding statistic in the dataset is difficult to obtain.

Power-law degree distributions: Degree distributions in each bipartite graph, which models inter-type interactions, are given in Fig. 9. Results show that all the degrees follow power-law distributions, which exactly meets the one in real networks.

Community evolution: To evaluate the performance of merge event modeled in Evolving K-graph, we assume that there are three communities C_1, C_2 and C_3 with fractions $\beta_1 = 0.3$, $\beta_2 = 0.4$ and $\beta_3 = 0.3$ at the initial time. Then, when the merge event starts we reset the parameters as $\beta_1 = 0.3$, $\beta_2 = 0$, $\beta_3 = 0$ and $\beta_4 = 0.7$ and merge the nodes in C_2 and C_3 into a new one C_4. To make the evaluation we use two community partitions, one is obtained through pre-set parameters, i.e., C_i, and the other is detected by Louvain algorithm [27] which is generated based on the network topology. We measure relative error (Err) between them and

the result is shown in Fig. 10. The figure shows that before the merge event, Err is approximately 0 which indicates that Evolving K-graph can well capture the community evolution. When the event happens, Err increases abruptly since that though we reset the community parameters, network topology can not change immediately and thus results in a dismatch between the two results. After that, Err decreases over time and finally reapproaches 0. Compare the top three figures where the merge event starts at $t = 100$, $t = 150$ and $t = 200$ respectively, and we could conclude that the network that starts merge later requires a larger merging time, which is consistent with our theoretical results. The same results appear in split event as shown in the three bottom figures. From the above results we demonstrate that Evolving K-Graph can well model the community structure and its evolution.

2) Comparison with the Baseline Model: In addition to the properties of data and Evolving K-Graph, Fig. 8 also presents that of Multi-typed Affiliation Network. The figure shows that, Multi-typed Affiliation Network can also well fit the real data points in terms of densification ratio and diameter. However, the positive correlation can only be reproduced in Evolving K-Graph while in Multi-typed Affiliation Networks it fluctuates around 0. In addition, we plot the degree distribution of the baseline model in Fig. 9 (b). Results show that it is power-law distributed. As for interactive evolution and community structure, the characterizations of them are innovations of the proposed model and the baseline model can not cover them, as shown in Table I. To sum up, Evolving K-Graph could better fit the dataset compared with the baseline one.

C. Parameters Choice and Difficulties in Fitting

We now present an illustration on the choice of parameters and the difficulties in fitting a specific dataset. All the involved parameters are listed in Table IV, which includes:

- $< p_i >$: Since p_i denotes the probability that a node of type i arrives the network, a larger p_i directly results in a larger number of nodes in V_i. Based on this we set $< p_1, p_2, p_3 >$ as $< 0.29, 0.38, 0.43 >$, which is in proportion to the final number of nodes as given in Table II.
- $< c_{ij} >$ and $< \hat{c}_{ij} >$: The parameter c_{ij} denotes the number of newly added edges and $\hat{c}_{ij}$ denotes that of newly leaved ones, and thus their combined effects determine the number of edges in the network. A larger c_{ij} and a smaller $\hat{c}_{ij}$ result in a larger number of edges.

However, unlike p_i that only affects nodes of type i, c_{ij} and $\hat{c}_{ij}$ have a cross-type effects and therefore bring difficulties in fitting. For example, a larger c_{ij} could cause a larger number of edges in $B(V_i, V_j)$, which, however, also results in a larger number of edges in $B(V_j, V_k)$ since $c_{ij}c_{ik}$ edges will be added in $B(V_j, V_k)$ due to interactive evolution. Therefore, the fitting of asymmetric networks whose properties in nodes of different types vary a lot, e.g., Academic Network in Table II, tends to be especially difficult to control. Further, note that the values of parameters sometimes vary with the time, e.g., the curves in Fig. 3 (b) should be fitted with time varying parameters.

VII. Conclusion and Future Work

In this paper, we propose a model named Evolving K-Graph to characterize evolving networks with hybrid interactions involving both inter-type connections and intra-type connections. The superiorities of Evolving K-Graph lie in three aspects: good capture of realistic networks, mathematical tractability and efficient implementation. Subsequently, we prove that the model can reproduce commonly-observed network properties such as power-law distribution, densification, shrinking diameter and community structure. And finally, the proposed model is verified through simulations and results show that it can well capture the realistic networks.

There remains some future directions that can be explored. For example, it is a desirable future work to study the network under shrinking mode, where the edge leave rate is faster than its creation rate. Another interesting work is to explore direct connections between nodes of a same type, which exist, since entities without common attributes may also related with each other, but are ignored in the current model.

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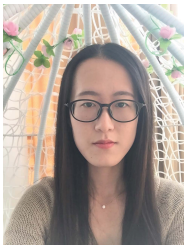

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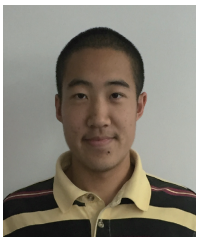

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